COMS 4281 - Introduction to Quantum Computing Fall 2024

Practice Worksheet #2

This practice worksheet is intended to cover material up to September 25. The weekly quiz (due September 29, 11:59pm) will be based on this worksheet. The midterm and final exam will have questions inspired by the worksheets.

Last week, we learned about the No-Cloning Theorem, entanglement, partial measurements, and measurements in other bases. We saw how to take the inner product, which is an important operation.

Problem 1: Inner products

Recall that $\langle\psi|\theta\rangle$ is the inner product between two vectors (row vector followed by a column vector), and this is a scalar. Simplify the following expressions.

(a) $\langle 1|0\rangle$

(b) $\langle -|+\rangle$

(c) $\langle +|1\rangle$

(d) $\langle 0,1|+,-\rangle$.

(e) $(\langle 1| \otimes \langle +|)(|+\rangle \otimes |0\rangle)$

(f) $\langle\phi|\psi\rangle$, where $|\phi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$ and $|\psi\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$

Problem 2: Entanglement

Recall that a two-qubit state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is *unentangled* (or, equivalently, a *product state*) if it can be written in the form of $|\varphi\rangle \otimes |\theta\rangle$ where $|\varphi\rangle, |\theta\rangle \in \mathbb{C}^2$ are single-qubit states. Otherwise, we call it *entangled*.

For the following states, show which ones are entangled, and which ones are not.

(a) $\frac{1}{2}\Big(|00\rangle - |01\rangle + |10\rangle + |11\rangle \Big)$

(b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$.

(c) $-\sqrt{\frac{1}{8}}|00\rangle + \sqrt{\frac{1}{24}}|01\rangle + \sqrt{\frac{13}{24}}|10\rangle - \sqrt{\frac{7}{24}}|11\rangle$

For states that have three or more qubits, entanglement gets more subtle. We say that an n-qubit state $|\psi\rangle$ is *completely unentangled* if it can be written as $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ where each of the $|\psi_i\rangle$'s are single-qubit states. Even if it's not completely unentangled, different groups of qubits can entangled with each other, and unentangled between groups. For example, in a 3-qubit state, the 1st qubit could be unentangled with the 2nd and 3rd, but those two are entangled with each other. States that only have one group of entangled qubits are called *completely entangled*.

In the following three qubit states, identify the groupings of qubits that entangled with each other, and unentangled between groups. Which states are completely unentangled, and which ones are entangled?

(a) $\frac{1}{\sqrt{8}}\Big(|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle \Big)$
(b) $\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle \Big)$
(c) $\frac{1}{\sqrt{2}}\Big(|000\rangle + |110\rangle \Big)$
(d) $\frac{1}{2}\Big(|000\rangle + |010\rangle - |101\rangle - |111\rangle \Big)$

Problem 3: Measurements in Non-Standard Bases

For each of the following states $|\psi\rangle$, (i) express the state in bases B. (ii) State the measurement statistics of measuring the state in the bases B:

(a) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the basis $B = (|+\rangle, |-\rangle)$.
(b) $|\psi\rangle = |0\rangle$ in the basis $B = (\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle)$.
(c) $|\psi\rangle = |-\rangle$ in the basis $B = (\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle)$.
(d) $|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$ in the basis $B = (\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle)$.

Problem 4: Partial Measurements

Consider performing a partial (standard basis) measurement on the first qubit of each of the following states. For each of the outcomes (either $|0\rangle$ or $|1\rangle$), what is (i) the unnormalized state after the partial measurement, (ii) the probability of that outcome, and (iii), the post-measurement state?

(a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
(b) $|\psi\rangle = \frac{1}{\sqrt{5}}|01\rangle + \sqrt{\frac{2}{5}}|10\rangle - \sqrt{\frac{2}{5}}|11\rangle$
(c) $|\psi\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|100\rangle + \frac{1}{2}|110\rangle$. Note that this is a three-qubit state. *Hint*: Before computing the partial measurement, what can you first observe about the state?

Now do (a)-(c) again, except the partial measurement is in the Hadamard basis $\{|+\rangle, |-\rangle\}$.

Problem 5: More Circuits Practice

(a) Write out the states $|\psi_n\rangle$ (corresponding to the intermediate states of the circuits after the dashed lines) in ket notation, for $n = 1, 2, 3$.

(b) In $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$, which groups of qubits are entangled with each other?

(c) Suppose we perform a partial measurement (in the standard basis) on the second and third qubits in $|\psi_3\rangle$. What is the distribution of outcomes, and what are the corresponding post-measurement states?