COMS 4281 - Introduction to Quantum Computing	Fall 2024

Practice Worksheet 3

This practice worksheet is intended to cover material up to October 7. The corresponding quiz (due October 11, 11:59pm) will be based on this worksheet. The midterm and final exam will have questions inspired by the worksheets.

Last week, we learned about the superdense coding, universal gate sets, the Solovay-Kitaev theorem, Deutsch's algorithm, and Simon's algorithm. After we learn about the Quantum Fourier Transform, we will combine our knowledge about quantum algorithms and the QFT to get Shor's algorithm.

Problem 1: Superdense Coding

Alice and Bob share an entangled Bell state, prepared by their friend Charlie.

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

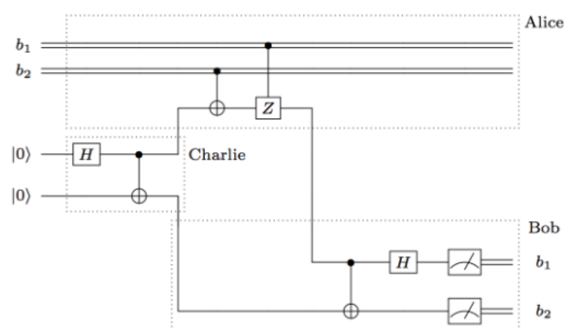

Figure 1: Classical bit transfer circuit.

Afterwards, Alice gets two bits b_1, b_2. Based on these bits (represented by the double lines) she applies gates to her qubit (represented by a single line) as depicted in the circuit.

(a) Suppose that $b_1 = 0, b_2 = 1$. What is the state of Alice's and Bob's qubits after Alice applies her operations, but *before* she sends her qubit to Bob (i.e., before he applies the CNOT and Hadamard at the bottom of the circuit)?

(b) What is the state of the two qubits *after* Bob applies his two gates?

Answer the two questions above, but for a general b_1, b_2.

Problem 2: Deutsch Algorithm

Recall the Deutsch problem from class:

Provided oracle access to $f : \{0,1\} \to \{0,1\}$, determine whether $f(0) = f(1)$ or $f(0) \neq f(1)$.

Classically, we must query f twice, once for $f(0)$ and once for $f(1)$. This is our first example of quantum speedup, as from the circuit given in Figure 2, we only require 1 query.

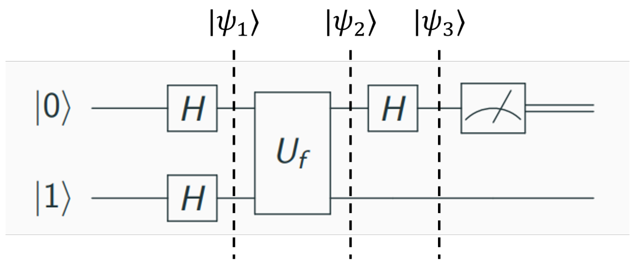

Figure 2: Deutsch Algorithm

With this problem, we will compute and show that only one query is needed to solve this problem.

(a) Compute $|\psi_1\rangle, |\psi_2\rangle$, and $|\psi_3\rangle$.

(b) Determine the measurement outcomes if

i. $f(0) = f(1)$

ii. $f(0) \neq f(1)$

Problem 3: Hadamard Practice

For the following input states, express the result in the standard basis after applying (i) $H^{\otimes 3}$ and (ii) $(HZH)^{\otimes 3}$. What do you notice about this second operation?

(a) $|000\rangle$

(b) $|010\rangle$

(c) $|101\rangle$

(iii) For $|a\rangle = |a_1 \ldots a_n\rangle$, show that

$$(HZH)^{\otimes n} |a\rangle = X^{\otimes n} |a\rangle .$$

This equality shows that a phase flip (Z) is equivalent to the bit flip (X) in the diagonal basis, a useful observation for Quantum Error Correction!

(iv) Now consider $|\psi\rangle = H^{\otimes 6} |110101\rangle$. If you were to write out $|\psi\rangle$ in the standard basis, there are $2^6 = 64$ terms! We don't want to make you do that (fortunately for you). However, please do compute the amplitude of the following basis states in $|\psi\rangle$:

- $|111111\rangle$
- $|001001\rangle$

Problem 4: Simon's Algorithm

Simon's Algorithm solves the following problem:

Given a function $f : \{0,1\}^n \to \{0,1\}^n$ that follows the property that for all $x \in \{0,1\}^n, f(x) = f(x \oplus s)$ for some $s \in \{0,1\}^n$, solve for s.

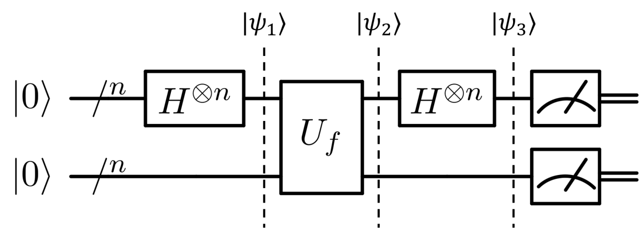

Figure 3: Simon's Algorithm

This problem directly inspired Shor to come up with his famous factoring algorithm!

Let $n = 3$, and the function $f : \{0,1\}^3 \to \{0,1\}^3$ be given by the following table.

x	$f(x)$
000	010
001	111
010	100
011	011
100	111
101	010
110	011
111	100

(a) Compute $|\psi_1\rangle, |\psi_2\rangle$, and $|\psi_3\rangle$. Recall that the way U_f works is as follows:

$$U_f |x, b\rangle = |x, b \oplus f(x)\rangle$$

where $x, b \in \{0,1\}^3$ and $b \oplus f(x)$ denotes the bitwise addition of b and $f(x)$.

(b) What is the probability distribution of the measurement outcomes?

Let's think about a 4-bit function $f : \{0,1\}^4 \to \{0,1\}^4$ now. However, we're not going to tell you what the truth table of f is, and so you can't immediately tell what the hidden shift s is (other than it's not the all zeroes string). Instead, imagine that you ran Simons' subroutine several times and got a particular sequence of measurement outcomes $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \ldots$. For each sequence, either solve for the hidden shift s or argue why the samples don't uniquely specify the hidden shift s.

(d)

$y^{(1)}$	0000
$y^{(2)}$	0000
$y^{(3)}$	1000
$y^{(4)}$	0100
$y^{(5)}$	1100

(e)

$y^{(1)}$	0110
$y^{(2)}$	1100
$y^{(3)}$	1010
$y^{(4)}$	1001
$y^{(5)}$	1111

(f)

$y^{(1)}$	0101
$y^{(2)}$	1010
$y^{(3)}$	1111
$y^{(4)}$	0111