COMS 4281 - Introduction to Quantum Computing Fall 2024

Practice Worksheet 5

This practice worksheet is intended to act as a review sheet for the midterm as well as give some practice for the more recently covered material. The midterm and final exam will have questions inspired by the worksheets.

Problem 1: Fourier Basis

Recall that the Fourier basis for $\mathbb{C}^N$ is comprised of the following states: for all $j = 0, 1, 2, \ldots N-1$,

$$|f_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(\frac{2\pi ijk}{N}\right) |k\rangle \ .$$

Remember, the phases $\exp(2\pi ijk/N)$ are all on the unit circle in the complex plane:

$$\exp\left(\frac{2\pi ijk}{N}\right) = \cos\left(\frac{2\pi jk}{N}\right) + i \sin\left(\frac{2\pi jk}{N}\right) \ .$$

(a) Write out the Fourier basis states for dimensions $N = 2, 3, 4$, where the phases are simplified as much as possible (don't just write them in terms of exponentials).

(b) Show that the Fourier basis states for $N = 4$ are (a) unit vectors and (b) orthogonal to each other. In other words, these vectors really do form a basis.

Problem 2: Fourier transform

Now recall the N-dimensional Fourier transform F_N, which is a unitary that maps the standard basis vector $|j\rangle$ to the j'th Fourier basis vector $|f_j\rangle$.

In what follows we'll set $N = 4$. Here, the standard basis states are $|0\rangle, |1\rangle, |2\rangle, |3\rangle$.

(a) Compute F_4 applied to $\frac{1}{\sqrt{4}}\Big(|0\rangle + |1\rangle + |2\rangle + |3\rangle\Big)$.

(b) Compute $F_4^\dagger$ (the inverse of F_4) applied to $\frac{1}{\sqrt{4}}\Big(|0\rangle + |1\rangle + |2\rangle + |3\rangle\Big)$.

(c) Compute F_6 applied to $\frac{1}{\sqrt{2}}\Big(|0\rangle + |3\rangle\Big)$.

(d) Compute F_6 applied to $\frac{1}{\sqrt{3}}\Big(|0\rangle + |2\rangle + |4\rangle\Big)$.

(e) Do you notice an interesting pattern with the previous two answers?

Problem 3: Fourier transform circuit

In class, we presented a recursive construction for F_N when N is a power of two (i.e. $N = 2^n$), expressed in terms of smaller DFTs $F_{N/2}$ and the "phase matrix" $A_{N/2}$.

Recall that $A_{N/2}$ is a $N/2 \times N/2$ matrix defined as follows:

$$A_{N/2} = \begin{pmatrix} 1 & & & & \\ & \omega_N & & & \\ & & \omega_N^2 & & \\ & & & \ddots & \\ & & & & \omega_N^{N/2-1} \end{pmatrix}$$

where $\omega_N = \exp(2\pi i/N)$.

(a) Write out A_2 in simplified terms (i.e., not in terms of exponentials). How many qubits does this matrix act on?

(b) Write out A_4 (it's okay to use exponentials). How many qubits does this matrix act on?

(c) Show that A_4 is also equal to $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.

(d) Draw the circuit for F_2.

(e) Use the recursive construction from class lecture as a reference, draw the QFT circuit for F_4, using only two-qubit and single-qubit gates. In your gate set, you're allowed to use arbitrary single-qubit unitaries (not only H, X, Z, etc.).

(f) Draw the QFT circuit for F_8.

Problem 4: Phase estimation

(a) What are eigenvector/eigenvalue pairs of the following unitary matrices? When writing the eigenvalue, write it in the form of $e^{2\pi i\varphi}$ for some $0 \leq \varphi < 1$.

i. X

ii. H

iii. $U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(b) For the unitary U above, one of the eigenvalues is $-i$. Let $|\psi\rangle$ denote the corresponding eigenvector from the previous subproblem. Write out the intermediate states of each of the phase estimation circuit with $t = 2$ ancilla qubits: