COMS 4281 - Introduction to Quantum Computing Fall 2024

Practice Worksheet 7

This worksheet covers topics from lectures on Nov 6, 11, and 13, including quantum complexity theory, spectral theorem, and Hamiltonians.

Problem 1: Quantum Complexity Theory

(a) Briefly explain the key differences between the complexity classes BQP and P.

(b) Is BPP $\subseteq$ BQP? Explain why or why not, and if it is unknown, explain known evidence.

(c) Is NP $\subseteq$ BQP? Explain why or why not, and if it is unknown, explain known evidence. (Hint: black box separation.)

(d) Explain why BQP $\subseteq$ PSPACE. Explain why PSPACE $\subseteq$ EXP.

(e) Explain how a quantum computer can solve 3SAT with n variables in $O(\sqrt{2^n} \cdot \mathrm{poly}(n))$ time. (Hint: Grover's search).

Problem 2: Grover amplification

Let $N = 2^n$. Let $f : \{0,1\}^n \to \{0,1\}$ be a function and let K denote the number of solutions x such that $f(x) = 1$. If K is known, then by running $O(\sqrt{N/K})$ Grover iterations one can obtain a solution with probability at least $1 - cK/N$ where c is a universal constant. However, if K is not known exactly, then by a more sophisticated algorithm (let's call it `ObliviousGrover`) one can still find a solution using $O(\sqrt{N/K})$ queries, but the probability of success is only guaranteed to be 99%.

For all ϵ, design an algorithm that calls `ObliviousGrover` as a subroutine and outputs a solution with probability $1 - \epsilon$. How many queries to f does your algorithm make, as a function of N, K, ϵ?

Problem 3: Spectral Theorem

Recall that the spectral decomposition of a $d \times d$ Hermitian matrix A is given by

$$A = \sum_{j=1}^{d} \lambda_j |v_j\rangle \langle v_j|$$

where $\{|v_j\rangle\}_j$ are the orthonormal basis states and λ_j are the corresponding eigenvalues.

(a) Consider the Pauli matrix $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

  i. Show that Y is Hermitian.
  ii. What are the eigenvalues of Y and the corresponding eigenvectors?
  iii. What is the spectral decomposition of Y?

(b) Repeat parts (a)-(c) for the Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(c) Let $f(x) = e^{-ix}$. What is the spectral decomposition of $f(Y)$?

Problem 4: Hamiltonians

Recall that for a Hamiltonian H, which can be diagonalized as $H = \sum_j E_j |v_j\rangle \langle v_j|$ where each $|v_j\rangle$ is an eigenstate with energy E_j. For a state $|\psi\rangle$, the energy with respect to H is $\langle\psi| H |\psi\rangle$.

(a) Consider the single-qubit Hamiltonian $H = X + Y$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

  i. What are its eigenstates and corresponding energies?
  ii. What is the energy of $|+\rangle$ with respect to this Hamiltonian?
  iii. What is the energy of $|0\rangle$ with respect to this Hamiltonian?

(b) Consider the two-qubit Hamiltonian $H = Z \otimes Z + X \otimes X$, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

  i. What are its eigenstates and corresponding energies?
  ii. What is the energy of $|+\rangle |0\rangle$ with respect to this Hamiltonian?
  iii. What is the energy of $|00\rangle$ with respect to this Hamiltonian?