Week 3: Entanglement, Teleportation, EPR Paradox

COMS 4281 (Fall 2024)

Admin

1. Practice problem sheet available, quiz on Gradescope. Quizzes should be done individually.
2. Pset1 due October 6, 11:59pm.
3. Use EdStem to find pset collaborators. **However you must write your own solutions**.

Upcoming Events

- This Sunday: NYC HAQ at Columbia.
  - Talks from JP Morgan, NVIDIA, and Wells Fargo about their quantum research (11 - 1pm, Davis Auditorium).
  - **Need a couple student volunteers to help!**
- **Seminar**: Quantum Science with Tweezer Arrays.
  - When, where: Monday, Sept 23, 12:30pm, Pupin 8th floor.
  - Who: Manuel Endres (Caltech).

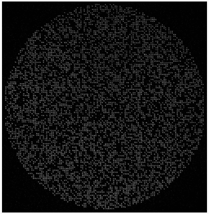
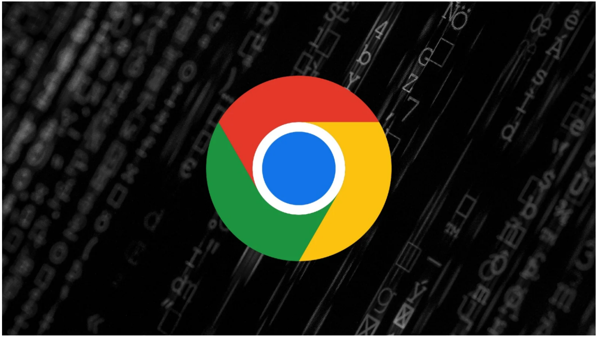


Figure 1: 6000 cesium atoms

In other news...

Chrome switching to NIST-approved ML-KEM quantum encryption

By **Bill Toulas** September 16, 2024 12:22 PM 2



Google is updating the post-quantum cryptography used in the Chrome browser to protect against TLS attacks using quantum computers and to mitigate store-now-decrypt-later attacks.

The upcoming change will swap Kyber used in hybrid key exchanges to a newer, and slightly modified version, renamed as Module Lattice Key Encapsulation Mechanism (ML-KEM).

Recap: Composite states

If the Hilbert space of one system is $\mathcal{H}_A$, and the Hilbert space of another system is $\mathcal{H}_B$, then the Hilbert space of them together is $\mathcal{H}_A \otimes \mathcal{H}_B$.

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A state of the joint system is a unit vector $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, which can be written as

$$|\psi\rangle = \sum_{\substack{1 \le i \le d_A \\ j \le j \le d_B}} \alpha_{ij} \, |i, j\rangle$$

where d_A, d_B are dimensions of $\mathcal{H}_A, \mathcal{H}_B$, respectively.

Recap: Composite states

Measuring $|\psi\rangle = \sum_{i,j} \alpha_{ij} |i,j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ yields

$$|i,j\rangle \text{ with probability } |\alpha_{ij}|^2$$

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Examples: What happens when we measure these two-qubit states?

- $|-\rangle \otimes |-\rangle$.
- $\frac{1}{\sqrt{6}} |0,0\rangle + \frac{1}{\sqrt{2}} |0,1\rangle + \frac{1}{\sqrt{3}} |1,1\rangle$.

Partial Measurements

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2. What should the state of the system be conditioned on a specific outcome?

Partial measurements

Example: Measuring second qubit of $|\psi\rangle = |-\rangle \otimes |-\rangle$.

- Distribution of outcomes:

- Post-measurement state given outcome $|b\rangle$:

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Since the two qubits are **unentangled**, measuring second qubit does not affect first qubit!

Partial measurements

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- Post-measurement state given outcome $|0\rangle$:

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- Post-measurement state given outcome $|1\rangle$:

$$\left(\sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle\right) \otimes |1\rangle$$

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Outcome distribution: The outcome distribution of measuring a single qubit should be consistent with the scenario when we measure *both* qubits.

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In other words, the following two distributions are the same:

1. Measuring qubit $\#2$.
2. Measuring both qubits, and taking the marginal distribution of the second qubit.

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Think of it as a quantum analogue of "Bayesian updating": when we learn the outcome of measuring the second qubit, the distribution of measuring the first qubit changes.

Partial measurements

Formal recipe for analyzing partial measurements on second qubit: first, re-arrange terms:

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |i, j\rangle = \sum_{j} \left(\sum_{i} \alpha_{ij} |i\rangle \right) \otimes |j\rangle$$

Partial measurements

Let $\beta_j = \sqrt{\sum_i |\alpha_{ij}|^2}$, then we can re-write the sum as

$$|\psi\rangle = \sum_j \beta_j \left(\sum_i \frac{\alpha_{ij}}{\beta_j} |i\rangle \right) \otimes |j\rangle \ .$$

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Note that $\sum_i \frac{\alpha_{ij}}{\beta_j} |i\rangle$ is a normalized quantum state on $\mathcal{H}_A$.

Partial measurements

Now, if we measure system $\mathcal{H}_B$, we'll see outcome $|j\rangle$ with probability

$$\beta_j^2 = \sum_i |\alpha_{ij}|^2$$

And the state of the first system after seeing $|j\rangle$ is

$$\frac{1}{\beta_j} \sum_i \alpha_{ij} |i\rangle \ .$$

Quantum Teleportation

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She could try to call Bob over the phone and send a classical description of $|\psi\rangle$, but that could require an infinite number of bits if $|\psi\rangle$ has amplitudes that use transcendental numbers.

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Suppose one year ago, Alice and Bob were in the same quantum lab, and they generated the entangled state

$$|\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle \Big) \,.$$

Alice takes the first qubit, Bob takes the second qubit and they go their separate ways.

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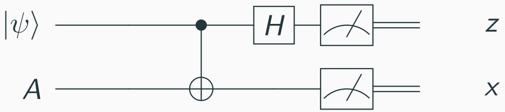
She performs the following circuit on her two qubits:



where A denotes Alice's half of the EPR pair, and the measurement outcomes are denoted $z, x \in \{0, 1\}$ respectively.

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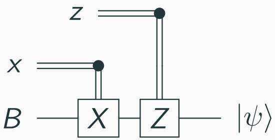
Alice tells Bob the bits z, x.

Quantum teleportation

Bob then applies the following gates to his EPR qubit B, depending on the values of x, z.



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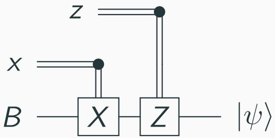
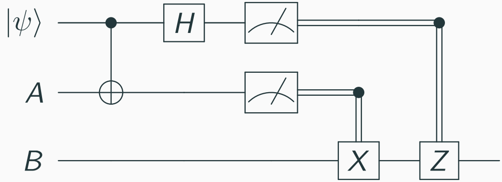


Recall that

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

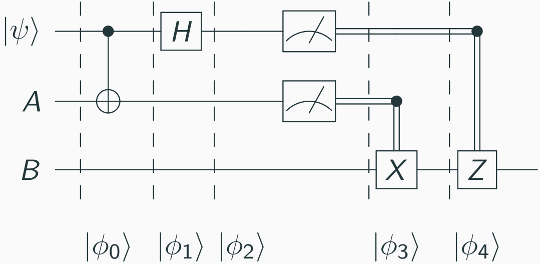
Quantum teleportation

Viewing the entire process as a circuit, this looks like



Quantum teleportation

We analyze it step by step:



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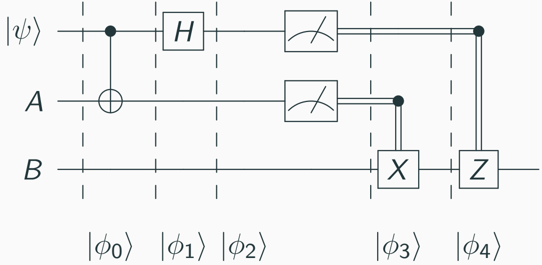

Let's do this together on the board!

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Both the preshared entanglement, and the classical communication are necessary for quantum teleportation to work.

Public Service Announcement

Entanglement does not allow for faster-than-light communication!

Nonstandard Measurements

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This is what we call a **standard basis measurement** or a **computational basis measurement**: the outcomes are the standard basis (a.k.a. computational basis) vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \cdots \qquad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Rewrite $|\psi\rangle$ as a linear combination of the $\{|+\rangle\,, |-\rangle\}$ basis:

$$|\psi\rangle = \alpha\frac{\sqrt{2}}{2}\Big(\,|+\rangle + |-\rangle\,\Big) + \beta\frac{\sqrt{2}}{2}\Big(\,|+\rangle - |-\rangle\,\Big)$$

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$$= \frac{\sqrt{2}}{2}\Big(\alpha + \beta\Big)\,|+\rangle + \frac{\sqrt{2}}{2}\Big(\alpha - \beta\Big)\,|-\rangle$$

Measuring in diagonal basis

The probability of getting $|+\rangle$ outcome is thus

$$\left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left|\alpha + \beta\right|^2$$

and similarly the probability of $|-\rangle$ outcome is

$$\left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left|\alpha - \beta\right|^2$$

Another perspective

There's also a geometric way to think about it:

$$\Pr\left[\text{measuring } |\psi\rangle \text{ in diagonal basis yields } |+\rangle\right] = \text{overlap of } |\psi\rangle \text{ and } |+\rangle$$

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Similarly,

$$\Pr\left[\text{measuring } |\psi\rangle \text{ in diagonal basis yields } |-\rangle\right] = \left|\left\langle - \,\middle|\, \psi \right\rangle\right|^2$$

General formula for measuring in a basis

Let $|\psi\rangle \in \mathbb{C}^d$ be a quantum state. Let $B = \{|b_1\rangle, \ldots, |b_d\rangle\}$ be an **orthonormal basis** for $\mathbb{C}^d$.

Then measuring $|\psi\rangle$ with respect to basis B yields outcome $|b_j\rangle$ with probability

$$\left|\left\langle b_j \middle| \psi \right\rangle\right|^2$$

and the post-measurement state is $|b_j\rangle$.

Implementing measurements in different bases

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Measuring $|\psi\rangle$ in a basis B is **equivalent** to

1. Apply a unitary V that maps B to standard basis (i.e. $|b_j\rangle \rightarrow |j\rangle$).
2. Measure in standard basis.

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1. Apply a unitary V that maps B to standard basis (i.e. $|b_j\rangle \to |j\rangle$).
2. Measure in standard basis.

Probability of getting $|j\rangle$ in this new process is the same as getting $|b_j\rangle$ when measuring in basis B.

Partial measurements plus nonstandard measurements

Let's combine the two concepts! Let $|\psi\rangle = \sum_{i,j} \alpha_{ij} |i\rangle \otimes |j\rangle$ denote a two-qubit state. Say we measure the first qubit with respect to basis $\{|b_0\rangle, |b_1\rangle\}$.

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Measurement rule: Probability of obtaining outcome $|b_0\rangle$ is the **length squared** of the vector

$$|a\rangle = (\langle b_0| \otimes I) |\psi\rangle = \sum_{i,j} \alpha_{ij} \langle b_0 \mid i\rangle \ |j\rangle .$$

Note this is a vector in $\mathbb{C}^2$, not $\mathbb{C}^2 \otimes \mathbb{C}^2$. It's a **partial inner product**.

Partial measurements plus nonstandard measurements

The probability is thus:

$$\Big\| \, |a\rangle \, \Big\|^2 = \sum_j \Big| \sum_i \alpha_{ij} \langle b_0 \mid i\rangle \Big|^2$$

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The **post-measurement** state of both qubits is

$$|b_0\rangle \otimes \frac{|a\rangle}{\| \, |a\rangle \, \|}.$$

Partial measurements plus nonstandard measurements

Let's work through an example:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \,.$$

Measure the **first** qubit in the **diagonal basis**.

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$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \,.$$

Measure the **first** qubit in the **diagonal basis**.

- What is the probability the outcome is $|+\rangle$? Or $|-\rangle$?
- What is the post-measurement state in either case?

Next time

Heisenberg's uncertainty principle, EPR paradox, Bell's Theorem.

Heisenberg Uncertainty Principle

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Popular science version: can't exactly know both the position and momentum of a particle simultaneously.

Heisenberg Uncertainty Principle

In quantum information theory terms: it is not possible for a qubit $|\psi\rangle \in \mathbb{C}^2$ to be simultaneously determined in both the **standard** basis and the **diagonal basis**.

Heisenberg Uncertainty Principle

In quantum information theory terms: it is not possible for a qubit $|\psi\rangle \in \mathbb{C}^2$ to be simultaneously determined in both the **standard** basis and the **diagonal basis**.

In other words, if measuring $|\psi\rangle$ in standard basis yields a deterministic outcome, then it **cannot** have a deterministic outcome if measured according to diagonal basis.

Heisenberg Uncertainty Principle

Important point: it's not about what happens if you **sequentially** measure the state $|\psi\rangle$ (what happens then?).

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It's reasoning about **counterfactual scenarios**: measuring $|\psi\rangle$ in the standard basis, **or** measuring $|\psi\rangle$ in the diagonal basis.

Heisenberg Uncertainty Principle

We say that the standard basis and diagonal basis are **incompatible** or **complementary**.

In quantum physics, the position and momentum of a particle correspond to incompatible measurements!