Week 6: The Quantum Fourier Transform

COMS 4281 (Fall 2024)

Admin

1. Practice worksheet and quiz #3 will be out tonight.
2. Midterm on October 21. More details soon.

Last time

- Quantum circuits
- Universal gate sets, Solovay-Kitaev Theorem
- Deutsch and Simons algorithms

Quantum Fourier Transform

Quantum algorithm that implements the Discrete Fourier Transform (DFT) on exponentially large dimension.

It is the heart of many powerful quantum algorithms such as

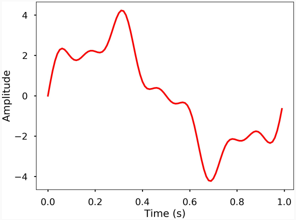
- Shor's factoring algorithm
- Phase estimation algorithm
- Algorithms for solving hidden subgroup problem

Discrete Fourier Transform

Discrete Fourier Transform

A method to uncover **hidden, periodic structure** in vectors. Used everywhere in engineering, science, and mathematics.

An example of a vector that represents a *noisy signal* whose characteristics we'd like to analyze.

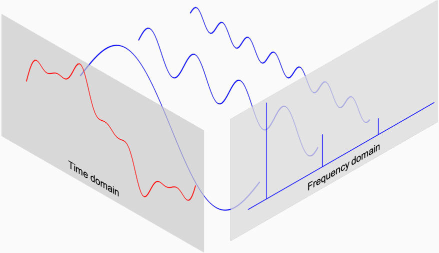


The entries of the vector correspond to equally spaced time points and the value of the entry corresponds to the signal amplitude at that time.

Discrete Fourier Transform

The **Discrete Fourier Transform (DFT)** is a method to express every vector (i.e. every signal) as a linear combination of simple periodic vectors (i.e. complex sinusoidal signals).

Visually:



Discrete Fourier Transform

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Mathematically: every vector $|\psi\rangle \in \mathbb{C}^N$ can be written as

$$|\psi\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j \, |f_j\rangle$$

where $\{|f_0\rangle, |f_1\rangle, \ldots, |f_{N-1}\rangle\}$ is the $\mathbb{Z}_N$**-Fourier basis**.

Fourier Basis

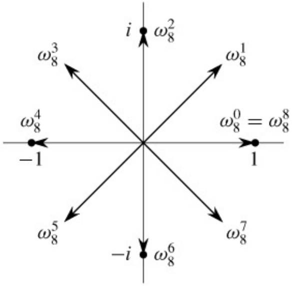
The Fourier basis vectors are

$$|f_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{jk} \, |k\rangle$$

for $j = 0, 1, \ldots, N-1$ where

$$\omega_N = \exp\Big(2\pi i/N\Big)$$

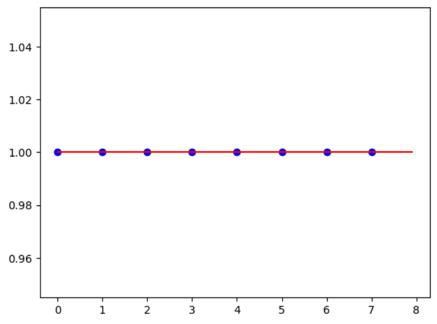
are the N'th roots of unity.



Fourier Basis

Example: $N = 8, j = 0$.

$$|f_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle = \frac{1}{\sqrt{N}} \Big(1, 1, \ldots, 1\Big)^{\top}$$

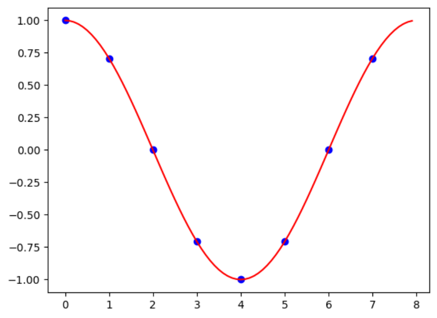


Fourier Basis

Example: $N = 8, j = 1$.

$$|f_1\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^k \, |k\rangle = \frac{1}{\sqrt{N}} \left(1, \omega_8, \omega_8^2, \ldots, \omega_8^7\right)^\top$$

Visually, the real component of ω_N^{jk}:

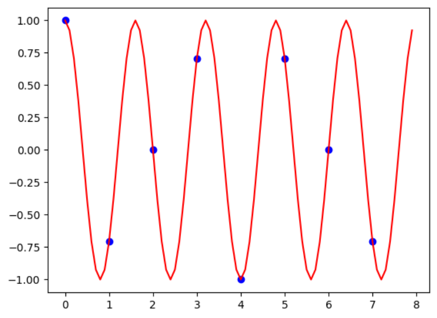


Fourier Basis

Example: $N = 8, j = 5$.

$$|f_5\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{5k} |k\rangle = \frac{1}{\sqrt{N}} \left(1, \omega_8^5, \omega_8^2, \omega_8^7, \dots\right)^\top$$

Visually, the real component of ω_N^{jk}:



- Let $|\psi\rangle = \sum_{j=0}^{N-1} \psi_j |j\rangle$ be a signal represented in the **standard basis**.
- In the Fourier basis, $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j \; |f_j\rangle$ for some **Fourier coefficients** $\hat{\psi}_j$.

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- The magnitude $|\hat{\psi}_j|^2$ of the j'th Fourier coefficient quantifies the contribution of $|f_j\rangle$ to $|\psi\rangle$.
- What's the relationship between the amplitudes $\{\psi_0, \psi_1, \ldots, \psi_{N-1}\}$ and the Fourier coefficients $\{\hat{\psi}_0, \hat{\psi}_1, \ldots, \hat{\psi}_{N-1}\}$?

They are related by a *unitary* transformation F_N (which is the DFT). It maps $|j\rangle \mapsto |f_j\rangle$. The *inverse DFT* is $F_N^\dagger$, which maps $|f_j\rangle$ to $|j\rangle$.

Applying $F_N^\dagger$ to $|\psi\rangle$ yields the vector

$$F_N^\dagger |\psi\rangle = |\hat{\psi}\rangle = \sum_{j=0}^{N-1} \hat{\psi}_j |j\rangle .$$

The **Fourier coefficients** of $|\psi\rangle$ have been turned into *amplitudes* in the standard basis of $|\hat{\psi}\rangle$.

Quantum Fourier Transform

Quantum Fourier Transform

Let $N = 2^n$. The **Quantum Fourier Transform (QFT)** is a quantum algorithm for implementing the n-qubit unitary F_N while taking only $\mathrm{poly}(n)$ time.

QFT maps quantum states $|\psi\rangle$ to their Fourier transform state $|\hat{\psi}\rangle = \sum_j \hat{\psi}_j |j\rangle$.

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Here, we associate $|j\rangle$ with the n-qubit state $|j_1 j_2 \cdots j_n\rangle$, the binary representation of j.

Quantum Fourier Transform

The best classical algorithm for computing DFT of a N-dimensional vector is called the **Fast Fourier Transform (FFT)**, which takes $O(N \log N)$ time.

Does this constitute an exponential speedup?

Quantum Fourier Transform

Not exactly. The Fast Fourier Transform gets an input that is a **classical** *N*-dimensional vector $|\psi\rangle$, and outputs another *N*-dimensional vector $|\hat{\psi}\rangle$.

On the other hand, QFT gets an input vector in **quantum form**, and produces an output vector in quantum form.

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On the other hand, QFT gets an input vector in **quantum form**, and produces an output vector in quantum form.

The Fourier coefficients $\{\hat{\psi}_j\}_j$ are not readily accessible: measuring $|\hat{\psi}\rangle$ produces basis vector $|j\rangle$ with probability $|\hat{\psi}_j|^2$, but afterwards all other information about the Fourier coefficients is lost.

The Fourier Transform unitary F_2

Example: $N = 2$ (single-qubit unitary)

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

We've seen this before...

The Fourier Transform unitary F_4

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$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} .$$

Can you spot recursive structure?

The Fourier Transform unitary F_4

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$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_2 & A_2 F_2 \\ F_2 & -A_2 F_2 \end{pmatrix}$$

where

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

The Fourier Transform unitary F_N

For general $N = 2^n$, if we order the columns where all the "even" columns are on the left, and all the "odd" columns are on the right, then

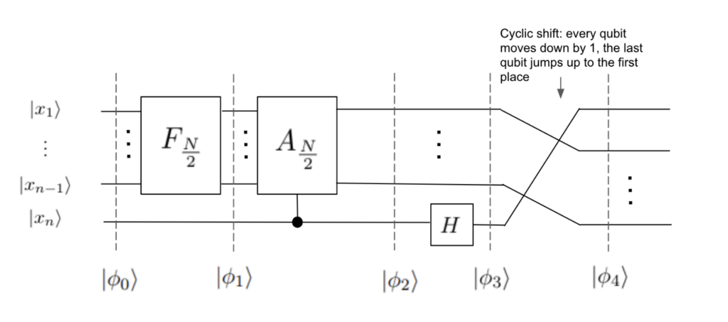
$$F_N = \frac{1}{\sqrt{2}} \begin{pmatrix} F_{\frac{N}{2}} & A_{\frac{N}{2}} F_{\frac{N}{2}} \\ F_{\frac{N}{2}} & -A_{\frac{N}{2}} F_{\frac{N}{2}} \end{pmatrix}$$

where

$$A_{N/2} = \begin{pmatrix} 1 & & & & \\ & \omega_N & & & \\ & & \omega_N^2 & & \\ & & & \ddots & \\ & & & & \omega_N^{N/2-1} \end{pmatrix}$$

The Fourier Transform circuit

The recursive formula for F_N inspires a recursive construction for the QFT circuit. Assume we already have circuits for the unitaries $F_{N/2}$ and $A_{N/2}$. Then the the circuit for F_N looks like



The Phase Circuit $A_{N/2}$

How is the circuit for $A_{N/2}$ implemented?

Note that the unitary acts on $n-1$ qubits, and for all $y \in \{0,1\}^{n-1}$ the unitary maps

$$|y_1, \ldots, y_{n-1}\rangle \mapsto \omega_N^{\mathrm{toint}(y)} |y_1, \ldots, y_{n-1}\rangle$$

where

$$\mathrm{toint}(y) = y_1 2^{n-2} + y_2 2^{n-3} + \cdots + y_{n-1}$$

The Phase Circuit $A_{N/2}$

Expanding and regrouping we get

$$\begin{aligned}
|y_1, \ldots, y_{n-1}\rangle &\mapsto \omega_N^{2^{n-2}\cdot y_1} \omega_N^{2^{n-3}\cdot y_2} \cdots \omega_N^{y_{n-1}} |y_1, \ldots, y_{n-1}\rangle \\
&= \left(\omega_N^{2^{n-2}\cdot y_1} |y_1\rangle\right)\left(\omega_N^{2^{n-3}\cdot y_2} |y_2\rangle\right) \cdots \left(\omega_N^{y_{n-1}} |y_{n-1}\rangle\right)
\end{aligned}$$

The Phase Circuit $A_{N/2}$

It is just a tensor product unitary!

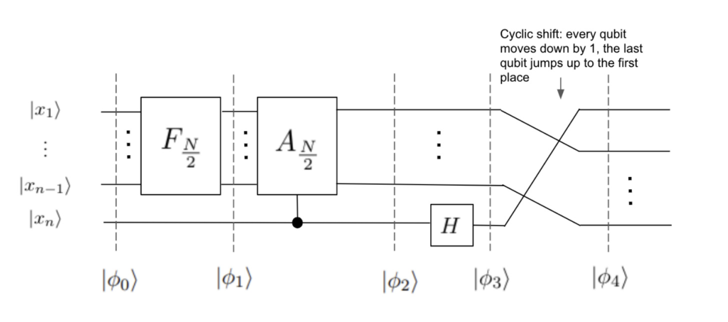
$$A_{N/2} = P(1/4) \otimes P(1/8) \otimes \cdots \otimes P(1/2^n)$$

where

$$P(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \varphi} \end{pmatrix}$$

The Fourier Transform circuit

Complexity analysis: Let $T(n)$ denote the number of gates used to construct F_N. It satisfies $T(n) = T(n-1) + O(n)$. Unrolling the recursion, we get $T(n) = O(n^2)$.



Analysis of the Quantum Fourier Transform circuit

Why does it work?

It's a couple pages of algebra... take a look at the scribe notes, or in Nielsen/Chuang Chapter 5 if you're interested!

Next time

Phase Estimation, the RSA Cryptosystem, and Shor's Algorithm.