Week 9: Quantum algorithms for search and counting

COMS 4281 (Fall 2024)

Admin

1. Practice worksheet, quiz out tonight.
2. Problem Set 2 out next week, due sometime before Thanksgiving.

So far

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- Simons Algorithm
- Order Finding/Factoring

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They all use a version of the Quantum Fourier Transform.

Unstructured search problem

Given: Oracle access to a black-box function $f : \{0,1\}^n \to \{0,1\}$

Goal: Output a **marked input** x such that $f(x) = 1$, if one exists.

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Classical algorithms need $\Omega(2^n)$ queries to f to find a marked input.

Grover's algorithm

Grover's Algorithm can find a marked input with $O(\sqrt{2^n})$ queries to f.

Presented by Lov Grover in 1997 in paper called "*A fast quantum mechanical algorithm for database search*".

Grover's algorithm

This achieves a **quadratic speedup**. Not as impressive as exponential speedup, but still interesting!

1 billion seconds $\approx$ 31 years

$\sqrt{1 \text{ billion}}$ seconds $\approx$ 9 hours.

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$$U_f \left|x, b\right\rangle = \left|x, b \oplus f(x)\right\rangle \ .$$

In Grover's algorithm the oracle is accessed via the **Phase oracle**

$$O_f \left|x\right\rangle = (-1)^{f(x)} \left|x\right\rangle .$$

Note that O_f acts on n **qubits**.

The Phase Oracle can be simulated by one query to the XOR oracle, and XOR oracle can be simulated with one query to the Phase Oracle.

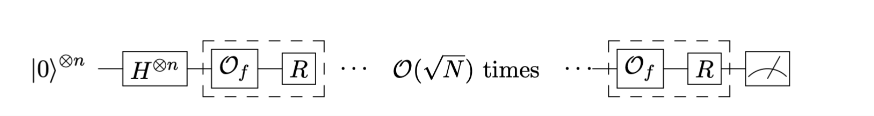
(do on board)

Grover's algorithm

Grover's algorithm

For simplicity, let's assume that there exists **exactly one** marked input.

The circuit for Grover's algorithm:



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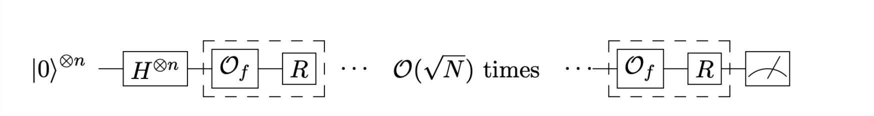
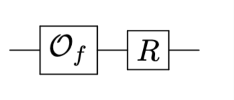


where



is the **Grover iterate**.

Grover iterate

The state of Grover's algorithm before the Grover iterates is

$$|+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \,.$$

The operator R is the n-qubit **Grover diffusion operator**

$$R = 2\,|+\rangle\,\langle+|^{\otimes n} - I$$

Exercise: check that this is unitary!

Analysis of Grover's algorithm

Let x^* denote the unique marked input.

Important fact: The intermediate states of Grover's algorithm are linear combinations of

$$|x^*\rangle \qquad \text{and} \qquad |\Delta\rangle = \frac{1}{\sqrt{2^n - 1}} \sum_{x \neq x^*} |x\rangle$$

We can prove this via induction.

Base case

Base case: initial state

$$|+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle = \sqrt{\frac{2^n - 1}{2^n}} |\Delta\rangle + \frac{1}{\sqrt{2^n}} |x^*\rangle$$

Inductive step

Assume that an intermediate state of Grover's algorithm has form $|\psi\rangle = \alpha |\Delta\rangle + \beta |x^*\rangle$.

Claim: $O_f |\psi\rangle$ is linear combination $|\Delta\rangle, |x^*\rangle$.

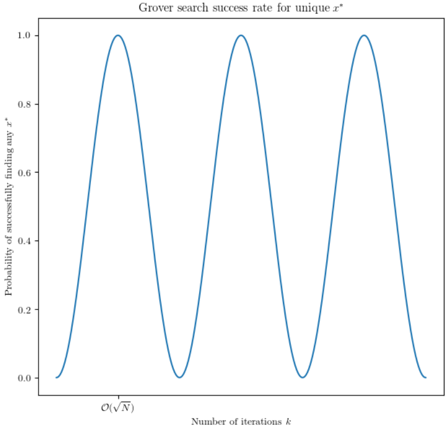
Claim: $R |\psi\rangle$ is linear combination of $|\Delta\rangle, |x^*\rangle$.

(prove Claims on board)

Analysis of Grover's algorithm

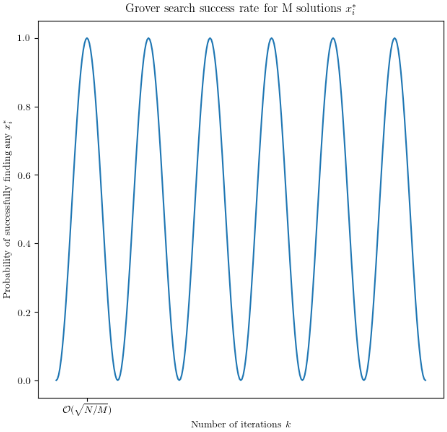
Success probability curve

The probability of finding a marked input when measuring the state after k Grover iterations. Running Grover's algorithm for longer can reduce your success probability!



Multiple solutions

What happens when there are $M > 1$ solutions to $f(x) = 1$? We can use Grover's algorithm as before, but the success probability curve changes:



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If the number of solutions M is **known**, then running $O(\sqrt{N/M})$ iterations will yield a solution with high probability.

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If M is **unknown**, by picking a random number T between 1 and $\sqrt{N}$ and stopping after T iterations, can find a solution with at least 50% probability. Repeating this $O(1)$ times yields a solution with high probability.

Multiple solutions

If M is **unknown**, then there is a more sophisticated solution to find a solution with $O(\sqrt{N/M})$ queries in expectation.

Idea: try Grover's algorithm with different number of iterations:

$$T_0, \lambda T_0, \lambda^2 T_0, \ldots$$

where T_0 is some constant and $1 < \lambda < 4/3$.

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For database search, the oracle $f(x)$ would correspond to something like "If Person x lives in New York City, has blood type O and likes Thai food, output 1" where $1 \leq x \leq N$.

Although Grover's algorithm makes $O(\sqrt{N})$ queries to f, computing $f(x)$ itself may have complexity $N = 2^n$.

Use cases for Grover search

A much better use is when we're trying to solve an abstract search problem where $f(x)$ can be computed much less than $N = 2^n$ time.

Example: Finding satisfying assignments to SAT formulas.

This is an NP complete problem. Believed best classical algorithms takes 2^n time.

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This is an NP complete problem. Believed best classical algorithms takes 2^n time.

With Grover, can solve it in $O(\sqrt{2^n}) \cdot \mathrm{poly}(n)$ time on a quantum computer.

Next time

Amplitude amplification and quantum counting.