Week 10: Quantum complexity theory

COMS 4281 (Fall 2024)

Quantum algorithms, so far

- Simons Algorithm
- Order Finding/Factoring, Phase Estimation
- Grover Search, Quantum Counting, Amplitude Amplification

Nearly 30 years after Shor's and Grover's algorithm, we still only have a very murky idea of when quantum computers are better than classical computers.

The formal study of this question is the focus of **quantum complexity theory**.

Computational complexity theory

Study of various **computational resources** needed to solve computational problems.

- Time
- Space
- Randomness
- Interaction
- Non-determinism
- Quantumness
- ...

Computational complexity theory

Main questions:

1. How do these computational resources relate to each other?
2. What are the tradeoffs?
3. Does non-determinism help speed up computations (P vs NP)
4. If a problem can be solved using a small amount of memory, can it also be solved using a small amount of time?
5. Can quantum computers efficiently solve problems that are hard for classical computers?

Complexity classes

Complexity classes are used to classify and compare computational problems according to the computational resources needed to solve them.

The focus is on **decision problems**, which are computational problems where for each **input** there is a **binary output** ("yes" or "no").

Example: Graph connectivity problem

- **Input**: graph G
- **Output**: is G connected?

Some complexity classes

P - polynomial time

Decision problems that can be solved by **deterministic** algorithms running in time $O(n^c)$ where n is the length of the input.

Traditionally considered the notion of **efficient classical computation** in theoretical computer science.

Examples: graph connectivity, determining if a number is prime, computing shortest paths in a graph

NP - nondeterministic polynomial time

Decision problems whose solutions can be **verified** in polynomial time. If the answer to the input is "yes", then there exists a solution/certificate/proof that is efficiently checkable.

Examples: traveling salesman person problem, boolean satisfiability, factoring.

Most optimization/search problems are in NP. Many are **NP-complete**, meaning they are amongst the hardest NP problems.

BPP - bounded-error probabilistic polynomial time

Decision problems that can be solved by a **randomized**, polynomial time algorithm.

The correct answer must be obtained with high probability (say 99%).

Examples: any problem in P, polynomial identity testing

It is conjectured that P = BPP (i.e. randomization does not help for efficient computation).

PSPACE - polynomial space

Decision problems that can be solved by a deterministic algorithm that uses $O(n^c)$ bits of space where n is the input length.

Captures the notion of problems that can be solved using a small amount of **memory**.

Examples: all of P, NP, BPP. Generalized tic-tac-toe, Super Mario Bros.

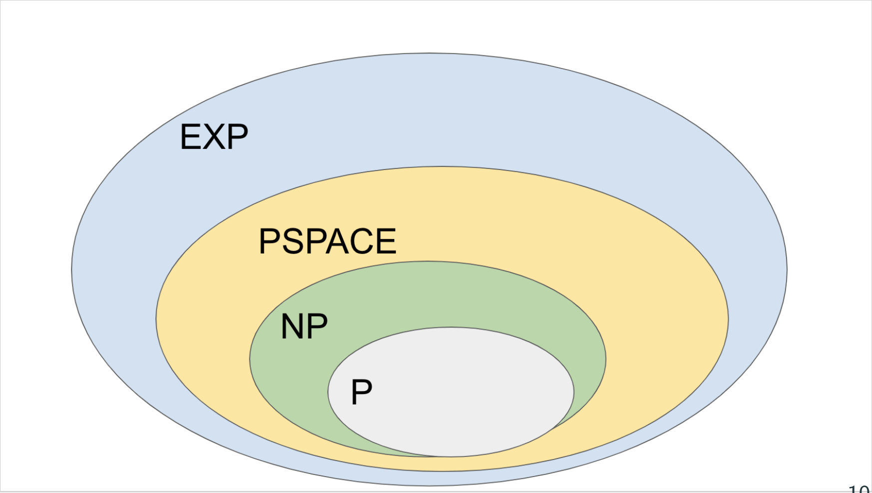
EXP - exponential time

Decision problems that can be solved by a deterministic algorithm that runs in $O(2^{n^c})$ time.

Examples: all of PSPACE, generalized Chess.

Classical complexity classes

The only separation we know is P $\neq$ EXP.



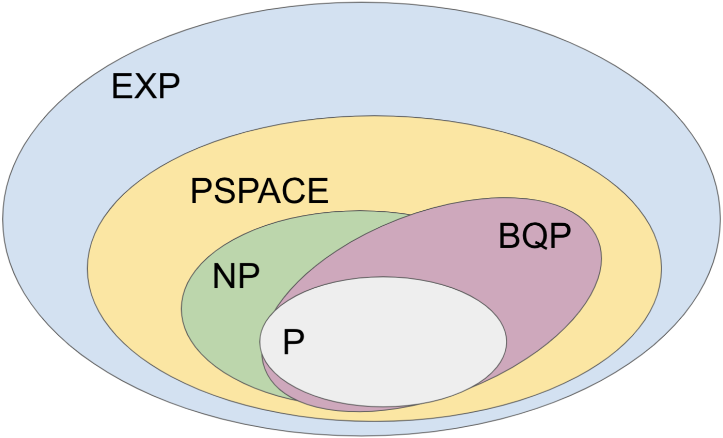
BQP - bounded-error quantum polynomial time

Decision problems that can be solved by a **quantum** algorithm (i.e. a quantum circuit) of $O(n^c)$ size with high probability.

Captures the notion of **efficient quantum computation**.

Examples: all of P, BPP, factoring, simulating quantum physics.

Where does BQP live?



The central questions

BQP $\subseteq$ **BPP**?

In other words, is there an efficient classical simulation of quantum computation?

NP $\subseteq$ **BQP**?

Can quantum computers be used to solve hard optimization problems like SAT or Traveling Salesperson Problem?

What are classical upper bounds on BQP?

What resources does a classical computer need in order to simulate quantum computations?

Acceptance probability of quantum circuit

Definition. The **acceptance probability** of a quantum circuit C on input $|\psi\rangle$ is the probability that measuring the first qubit of $C|\psi\rangle$ yields $|1\rangle$ (i.e. accepts).

APPROX-Q-CIRCUIT

Define the decision problem *APPROX-Q-CIRCUIT*:

Input: A description of quantum circuit C where either

1. $\Pr[C\,|0\cdots 0\rangle \text{ accepts }] \geq .99$
2. $\Pr[C\,|0\cdots 0\rangle \text{ accepts }] \leq .01$

Output: Determine which is the case.

APPROX-Q-CIRCUIT is a **promise problem** because the input is promised to satisfy some condition.

APPROX-Q-CIRCUIT

APPROX-Q-CIRCUIT is the canonical **BQP complete** problem, meaning that it is the "hardest" problem in BQP. If problem A is in BQP, then it can be reduced to an instance of APPROX-Q-CIRCUIT.

In other words, if there is a fast classical algorithm for APPROX-Q-CIRCUIT, then that can be used to solve any problem in BQP.

Claim: APPROX-Q-CIRCUIT is solvable in BQP.

Proof: The description of the circuit C is a list of single- and two-qubit gates $g_1, g_2, \ldots, g_T$ acting on various qubits.

The quantum algorithm to solve APPROX-Q-CIRCUIT is almost tautological: run the gates $g_1, g_2, \ldots$ in sequence, and measure the first qubit of resulting state.

This takes linear time in the size of the circuit C.

Exponential-time upper bound

APPROX-Q-CIRCUIT $\in$ EXP

The acceptance probability of a quantum circuit can be computed by a classical computer in exponential time.

Proof: Do what we've been doing in class: to compute the result of a circuit, compute the classical description of the state after applying a gate:

$$|\psi_{t+1}\rangle = g_{t+1} |\psi_t\rangle$$

The matrix-vector multiplication takes $(2^n)^2)$ time (if $n =$ number of qubits of C). Doing this T times requires $O(T \cdot 2^{2n})$ time.

APPROX-Q-CIRCUIT $\in$ EXP

If $|\psi_T\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$, then

$$\Pr\left[C|0\cdots0\rangle \text{ accepts}\right] = \sum_{x\in\{0,1\}^n : x_1 = 1} |\alpha_x|^2$$

Since APPROX-Q-CIRCUIT is BQP-complete, this means BQP $\subseteq$ EXP.

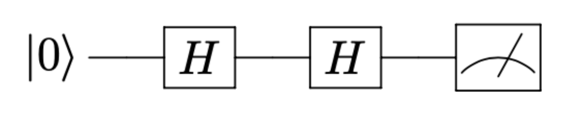
Polynomial-space upper bound

APPROX-Q-CIRCUIT $\in$ PSPACE

Quantum computations can also be classically simulated using **polynomial space**. This is based on the **sum-over-histories** or **Feynman path integral** approach.

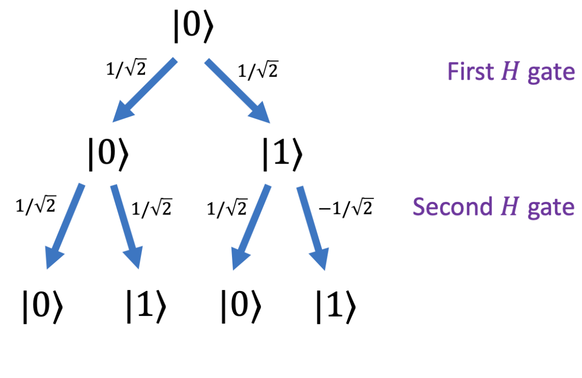
Key idea: in polynomial space, can iterate over exponentially many possible "histories" or "paths" of a quantum computation, and add up their amplitudes to determine final probability of measuring $|1\rangle$ in output qubit.

Simple circuit example:



Sum-over-histories approach

The amplitudes of the output state can be calculated via a tree:



Final amplitude of $|b\rangle$ = sum of amplitudes of all paths from $|0\rangle \rightarrow |b\rangle$.

Sum-over-histories approach

More generally, suppose we have gates $g_1, g_2, g_3, \ldots, g_T$ in a n-qubit circuit.

The computation can be represented by a tree where:

1. root node is labelled by $|0 \cdots 0\rangle$
2. each node has 2^n children labeled by $|x\rangle$ for $x \in \{0,1\}^n$
3. edge from node $|x\rangle$ in layer t to node $|y\rangle$ in layer $t+1$ is labeled by **transition amplitude** $\langle y| g_t |x\rangle$

Transition amplitudes

Claim: Given n-qubit basis states $|x\rangle, |y\rangle$, and two-qubit gate g, the transition amplitude $\langle y| g |x\rangle$ can be computed in (classical) polynomial time.

Proof: Assume without loss of generality that g acts on first two qubits. Then we are really calculating

$$\begin{aligned}&\langle y_1 y_2 \cdots y_n| (g \otimes I_{n-2}) |x_1 x_2 \cdots x_n\rangle \\ &\quad = \langle y_1 y_2| g |x_1 x_2\rangle \cdot \langle y_3 y_4 \cdots y_n \mid x_3 x_4 \cdots x_n\rangle\end{aligned}$$

where g is a 4×4 unitary matrix and I_{n-2} is identity on $n-2$ qubits.

Note: $\langle y_1 y_2| g |x_1 x_2\rangle$ is the entry of g in row indexed by (y_1, y_2) and column indexed by (x_1, x_2).

Subroutine: ComputeAmp

Input: basis states $|x\rangle, |y\rangle$

1. $amp \leftarrow 0$
2. For $u_1, u_2, \ldots, u_{T-1} \in \{0,1\}^n$:
    2.1 $amp \mathrel{+}= \langle y| g_T |u_{T-1}\rangle \langle u_{T-1}| g_{T-1} |u_{T-2}\rangle \cdots \langle u_1| g_1 |x\rangle$
3. Return amp

Subroutine: ComputeAmp

Complexity analysis: The subroutine *ComputeAmp* computes the overall transition amplitude $\langle y| C |x\rangle$.

It requires $O(Tn)$ bits of space to store $u_1, \ldots, u_{T-1}$ and $\mathrm{poly}(n)$ bits to store *amp*.

It takes $2^{O(Tn)}$ time to loop over all possible paths $(u_1, \ldots, u_{T-1})$.

For each path, updating the amplitude takes polynomial time (because $\langle u_j| g_j |u_{j-1}\rangle$ is computable in polynomial time).

Polynomial space algorithm for APPROX-Q-CIRCUIT

Input: Quantum circuit C satisfying promise

1. $prob \leftarrow 0$
2. For all $y \in \{0,1\}^n$ where $y_1 = 1$:
   2.1 $prob \mathrel{+}= |ComputeAmp(0^n, y)|^2$
3. If $prob \geq .99$ then output **YES**, otherwise output **NO**

The space usage of this algorithm is $\mathrm{poly}(n)$ (to store y and *prob*) plus whatever *ComputeAmp* needs, which is polynomial space.

It computes probability of getting $|1\rangle$ in first qubit of final state.

Next time

BQP vs NP, lower bounds on Grover search, random circuit sampling.