Quantum Error Correction

Admin

Dec 4 !!

- Pset 2 (Theory) due ~~November 22~~
- The last quiz will be a practice final
- In-class final on December 9
- Pset 2 (Coding) due December 15

Quantum Computing Inc tour: Dec 11.

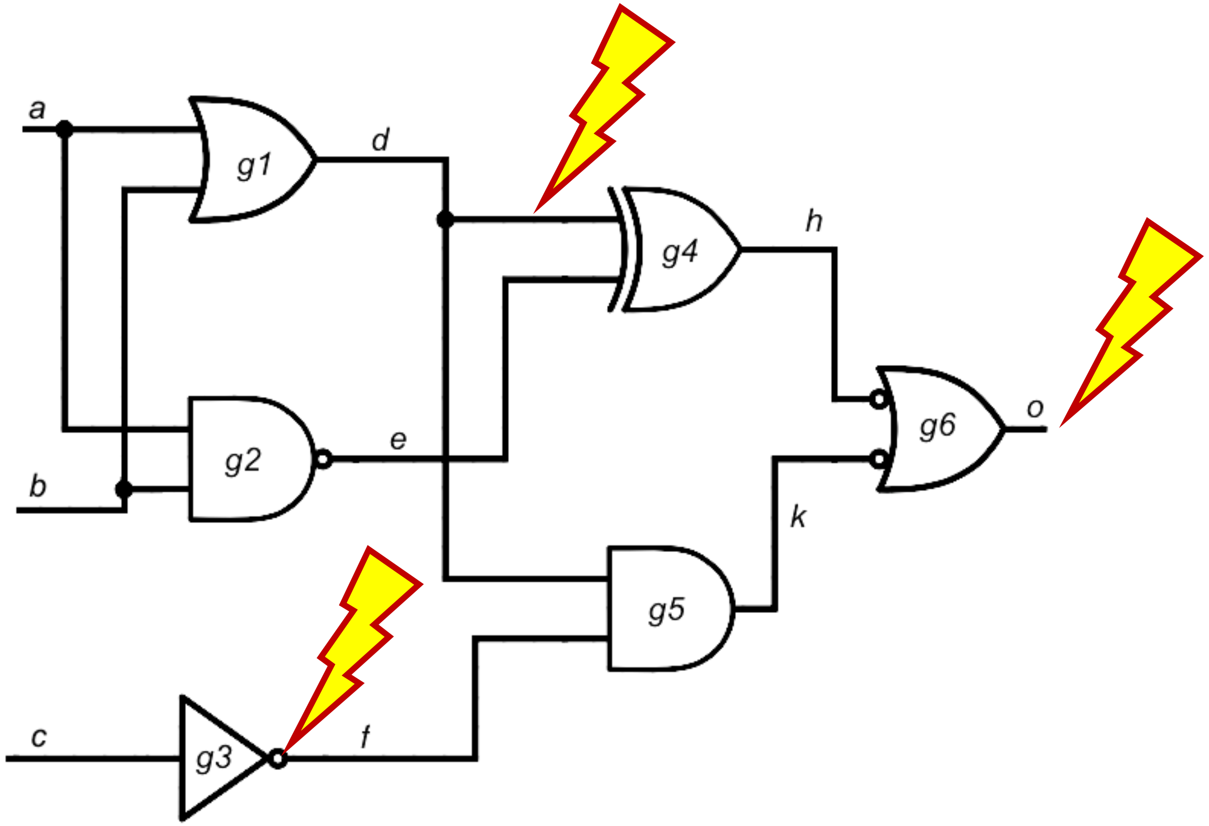
Motivation

- The biggest barrier to building a quantum computer capable of running the algorithms we learned about is **noise**.

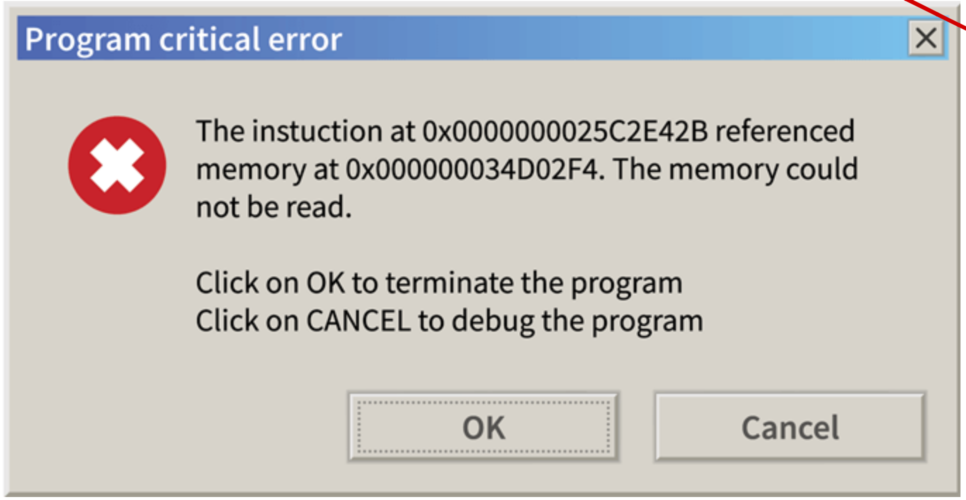
- Noise comes from many sources
  - Imperfections in device manufacturing
  - Interactions between device and environment
  - Imperfections in gate operations (over-rotation, mis-alignment, power fluctuations, timing errors,….)
  - Unwanted interactions between qubits (cross-talk)
  - ….

- Quantum systems are more susceptible to noise than classical systems are!

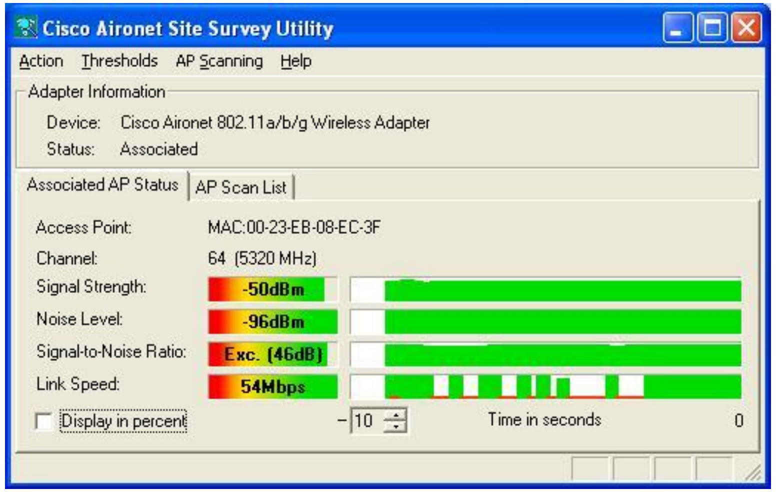
Errors in classical computation

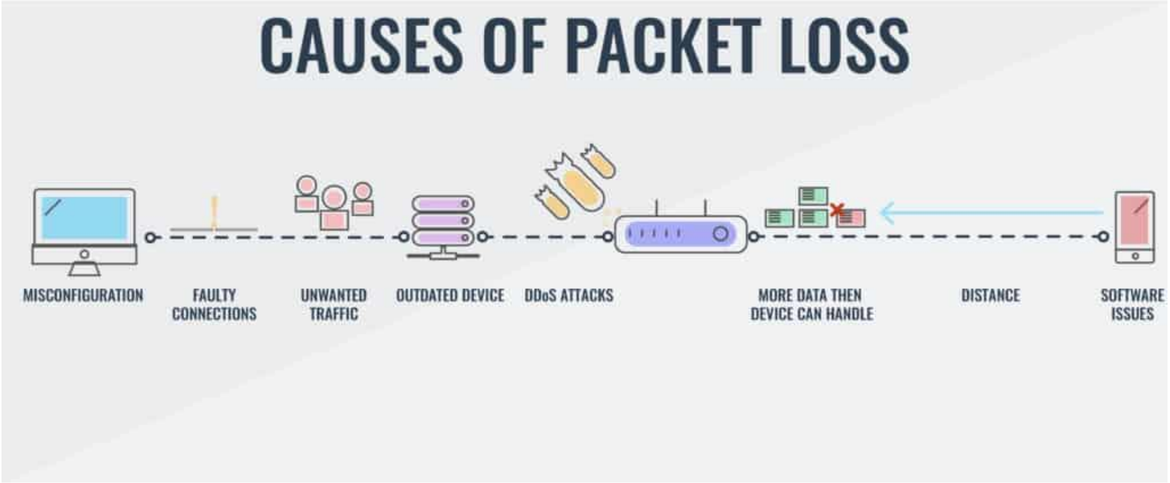


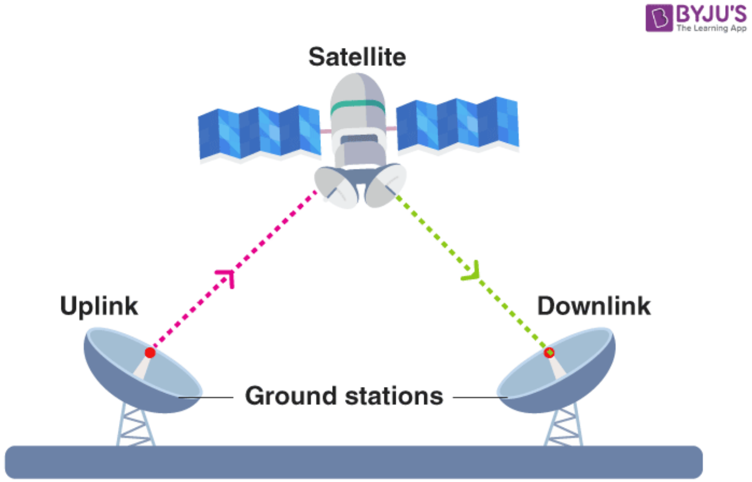
Errors in classical information storage



Errors in classical communication







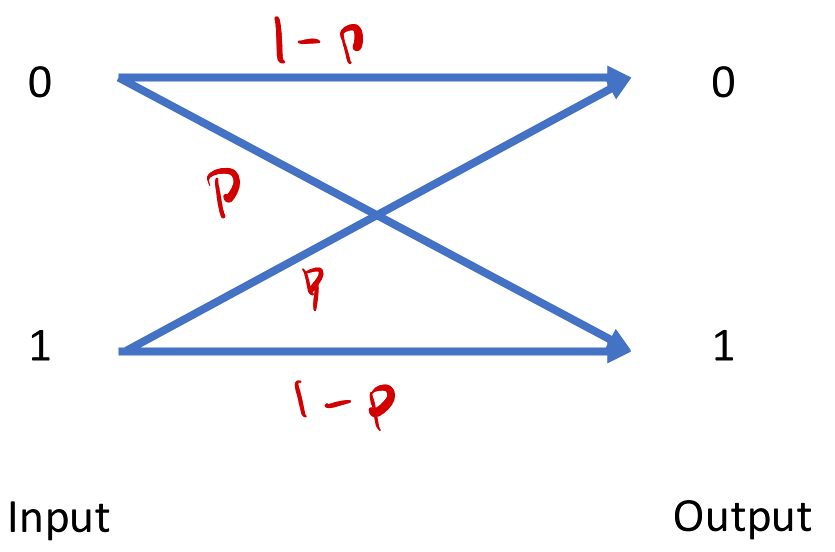
Solution: redundancy

- Storage and transmission: error-correcting codes
    - Wifi/3G/4G: LDPC codes, convolutional codes (Turbo Codes, etc)
    - CD-ROM: Reed-Solomon codes

- Computation: repetition and majority decoding
    - Repeat every computation 3 times and take majority vote
    - Von Neumann, "*Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components*", 1947.
    - Used in computers in spacecraft/high radiation environments

(Classical) error models

Simplest error model: bit flip channel

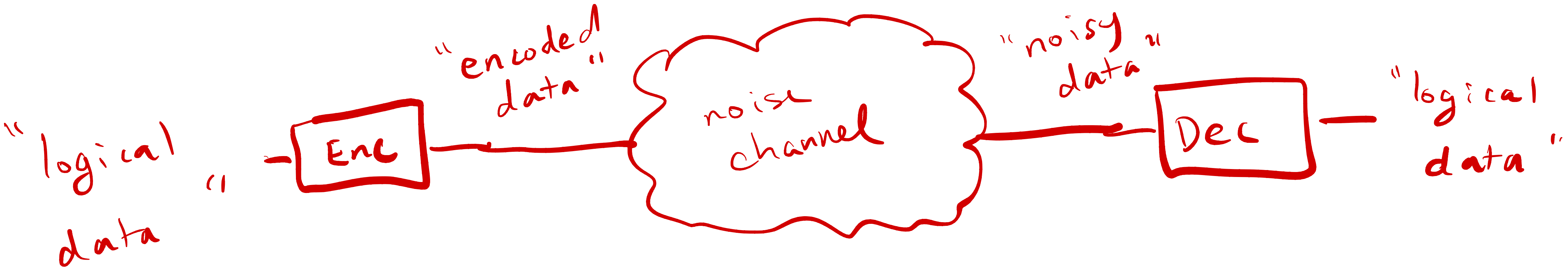


An n-bit string will experience an error with overwhelming probability.

$$\Pr[\text{n-bit string experiences no error}] = (1-p)^n = \exp(-\Omega(n))$$

(Classical) error models

General error model



(Classical) error correcting codes

Simplest code: repetition code

physical bits (aka encoded bits)

logical bits

$$0 \mapsto \underbrace{000\cdots 0} = \bar{0}$$

$$1 \mapsto 111\cdots 1 = \bar{1}$$

Decoding: take majority

Consider the i.i.d. bit flip channel with noise rate $p \ll \frac{1}{2}$.

$$\Pr[\text{decoding is wrong}] = \Pr[\text{more than } \tfrac{n}{2} \text{ bits get flipped}]$$

$$\leq \sum_{S \subseteq [n] : |S| \geq \frac{n}{2}} p^{|S|}(1-p)^{n-|S|} \ll \exp(-\Omega(n)).$$

(Classical) error correcting codes

A huge zoo of error-correcting codes, with different properties:

- Hamming codes
- Reed-Solomon/Reed-Muller codes
- LDPC codes
- Polar codes
- Expander codes
- …

Errors in (classical) computations

- Error rates in modern computers are *extremely* low.

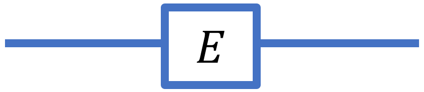
- A bit flip error rate $p < 10^{-16}$.

- Only very basic error detection used (there is a parity check bit for every word).

- Von Neumann's scheme is not really used (it was more relevant for the vacuum tube era)

Errors in the quantum realm

- **Problem #1:** There seems to be an infinite number of possible error, even for a single qubit!



- **Problem #2**: cannot copy quantum states!

not possible.

$$|\psi\rangle \not\to |\psi\rangle|\psi\rangle\cdots|\psi\rangle$$

- **Problem #3**: measurement destroys quantum information.

- In the early 90s, people thought these issues spelled doom for quantum computers.

Insight: quantum errors can be digitized

Can reduce problem of error-correction for a single-qubit to a **discrete** set of errors

- Bitflip error: $X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle$

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle.$$

- Phaseflip error:

$$Z|0\rangle = |0\rangle \qquad Z|1\rangle = -|1\rangle$$

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

- Bitflip & Phaseflip error:

$$XZ(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle - \beta|0\rangle$$

$$ZX(\alpha|0\rangle + \beta|1\rangle) = -\alpha|1\rangle + \beta|0\rangle.$$

Correcting bit flip errors

Use repetition code

$$|0\rangle \mapsto |000\rangle = |\bar{0}\rangle$$

$$|1\rangle \mapsto |111\rangle = |\bar{1}\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle = |\bar{\psi}\rangle$$

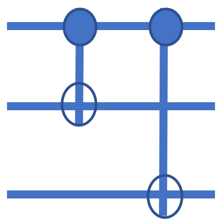
notation for encoded states.

Logical state

$|\psi\rangle$

$|0\rangle$

$|0\rangle$



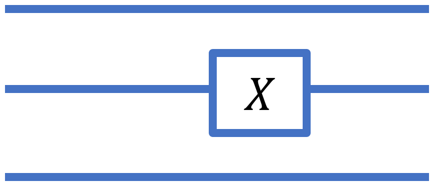
Encoded state

Enc

Note: this does not violate the no-cloning principle, because we're only copying in the standard basis.

Detecting and correcting bit flip errors

$|\overline{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$



$\alpha|010\rangle + \beta|101\rangle = |\tilde{\psi}\rangle$

- Directly measuring the encoded qubits would destroy the superposition in the logical state.

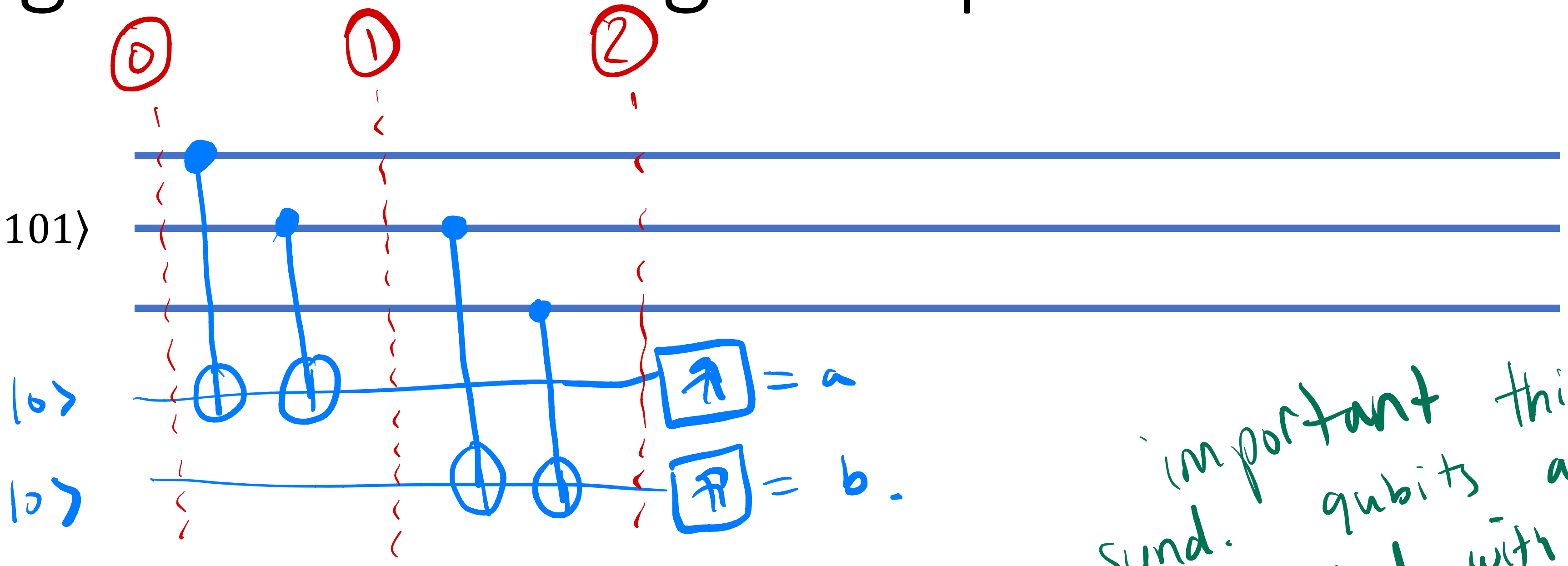
If measure $|\tilde{\psi}\rangle$, get $|010\rangle$ w.p. $|\alpha|^2$, $|101\rangle$ w.p. $|\beta|^2$,

- Instead, compute ***error syndromes*** and measure *those*.

but lost $\alpha|0\rangle + \beta|1\rangle$!

Detecting and correcting bit flip errors

$|\tilde{\psi}\rangle = \alpha|010\rangle + \beta|101\rangle$



syndrome qubits

important thing: synd. qubits are unentangled with phys qubits.

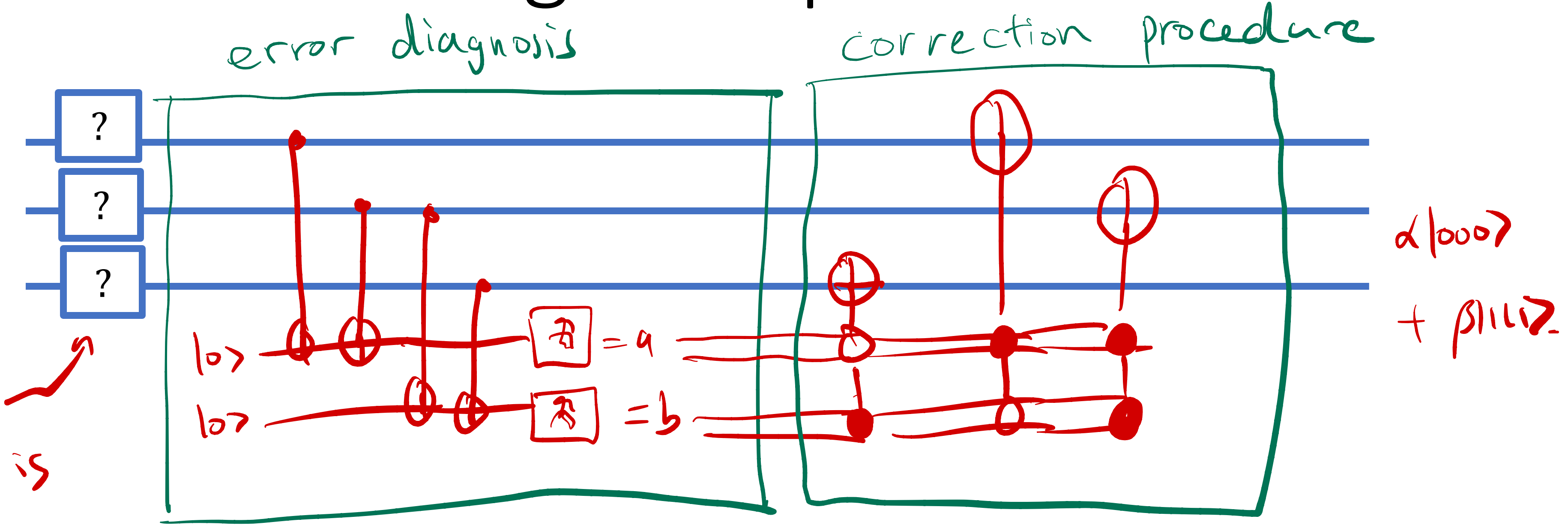
(0) $|\tilde{\psi}\rangle|00\rangle = \alpha|010\rangle|00\rangle + \beta|101\rangle|00\rangle$

(1) $\alpha|010\rangle|10\rangle + \beta|101\rangle|10\rangle$

(2) $\alpha|010\rangle|11\rangle + \beta|101\rangle|11\rangle = (\alpha|010\rangle + \beta|101\rangle)|11\rangle$

Detecting and correcting bit flip errors

$|\bar{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$



Assume: at most one of them is X

a	b	error
0	0	no error
0	1	3rd qubit
1	0	1st qubit
1	1	2nd qubit

To recover original logical qubit $|\psi\rangle$, just run encoding procedure in reverse.

Correcting ***phase*** flip errors

key observation:

phase flip = bit flip in diagonal basis.

$Z|+\rangle = |-\rangle$

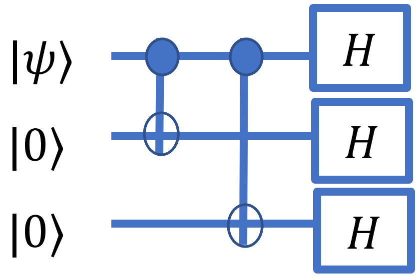
$Z|-\rangle = |+\rangle.$

Use repetition code in ***diagonal*** basis!

$$|0\rangle \mapsto |{+}{+}{+}\rangle$$

$$|1\rangle \mapsto |{-}{-}{-}\rangle$$

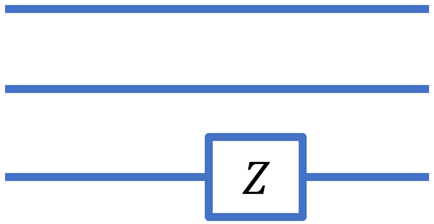
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle =$$



Enc

Detecting and correcting phase flip errors

$$|\overline{\psi}\rangle = \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$$

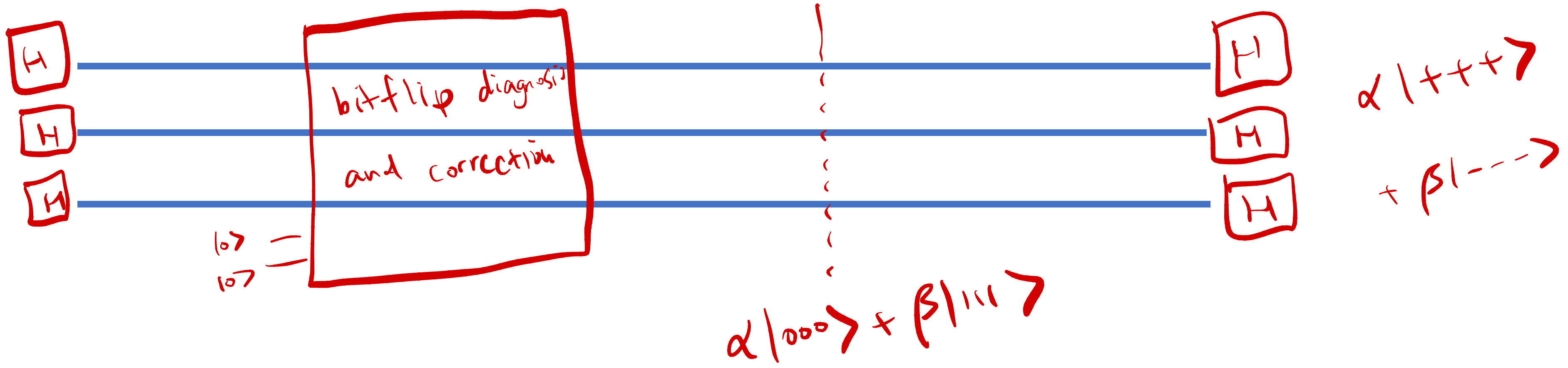


$$|\tilde{\psi}\rangle = \alpha|{+}{+}{-}\rangle + \beta|{-}{-}{+}\rangle$$

$\cancel{\alpha}$

Detecting and correcting phase flip errors

$$|\tilde{\psi}\rangle = \alpha|{+}{+}{-}\rangle + \beta|{-}{-}{+}\rangle$$



key idea: use bit flip as a black box,
but rotate from diagonal basis at
beginning (and return to diagonal basis at
the end).

Detecting and correcting phase flip errors

$|\tilde{\psi}\rangle = \alpha|{+}{+}\,{-}\rangle + \beta|{-}{-}\,{+}\rangle$

Shor's 9 qubit code

Concatenate bitflip and phaseflip codes together

phaseflip encoding.

$$|0\rangle \mapsto |+\rangle^{\otimes 3} \mapsto \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)^{\otimes 3} = \frac{1}{\sqrt{8}}\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$

logical qubits

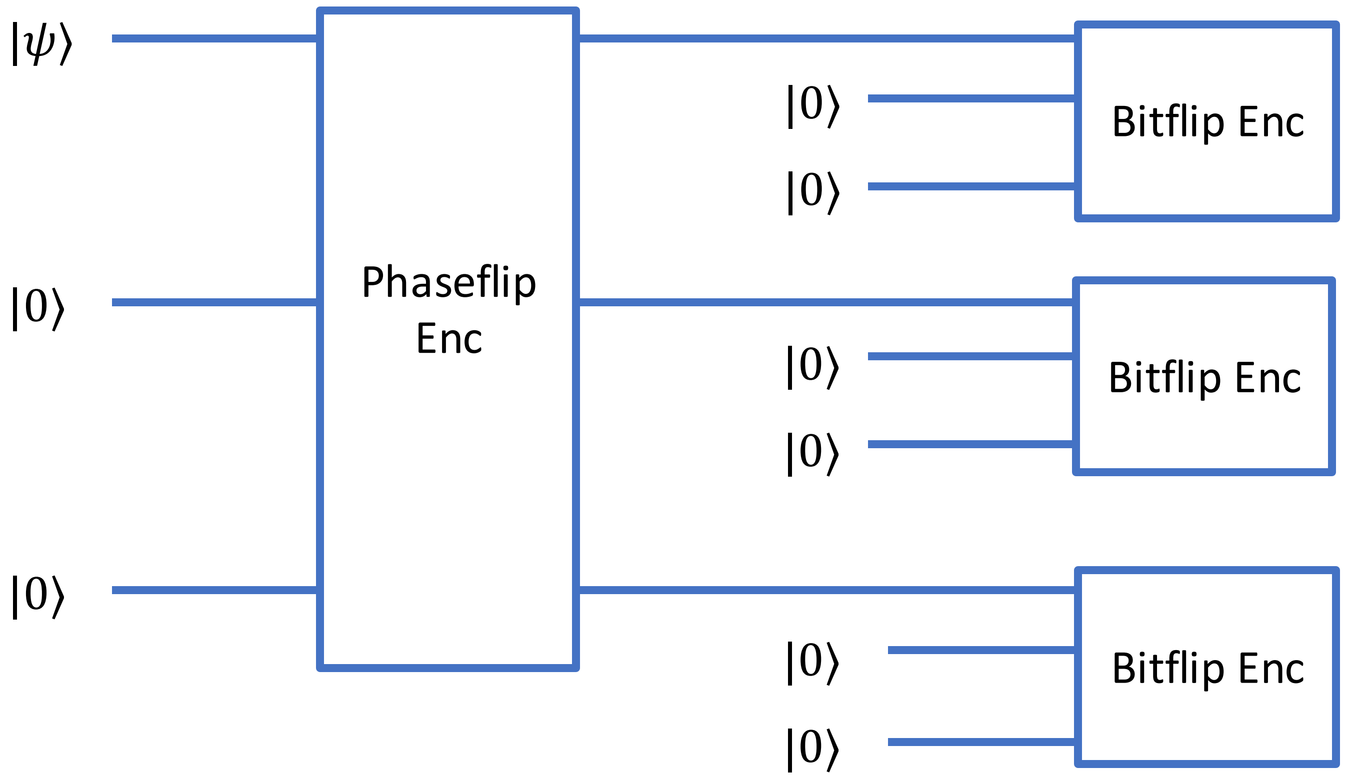
$$|1\rangle \mapsto |-\rangle^{\otimes 3} \mapsto \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)^{\otimes 3}$$

bit flip encoding.

Claim: this code can correct both bitflip and phaseflip errors, so long as they occur on a single qubit.

Shor’s 9 qubit code

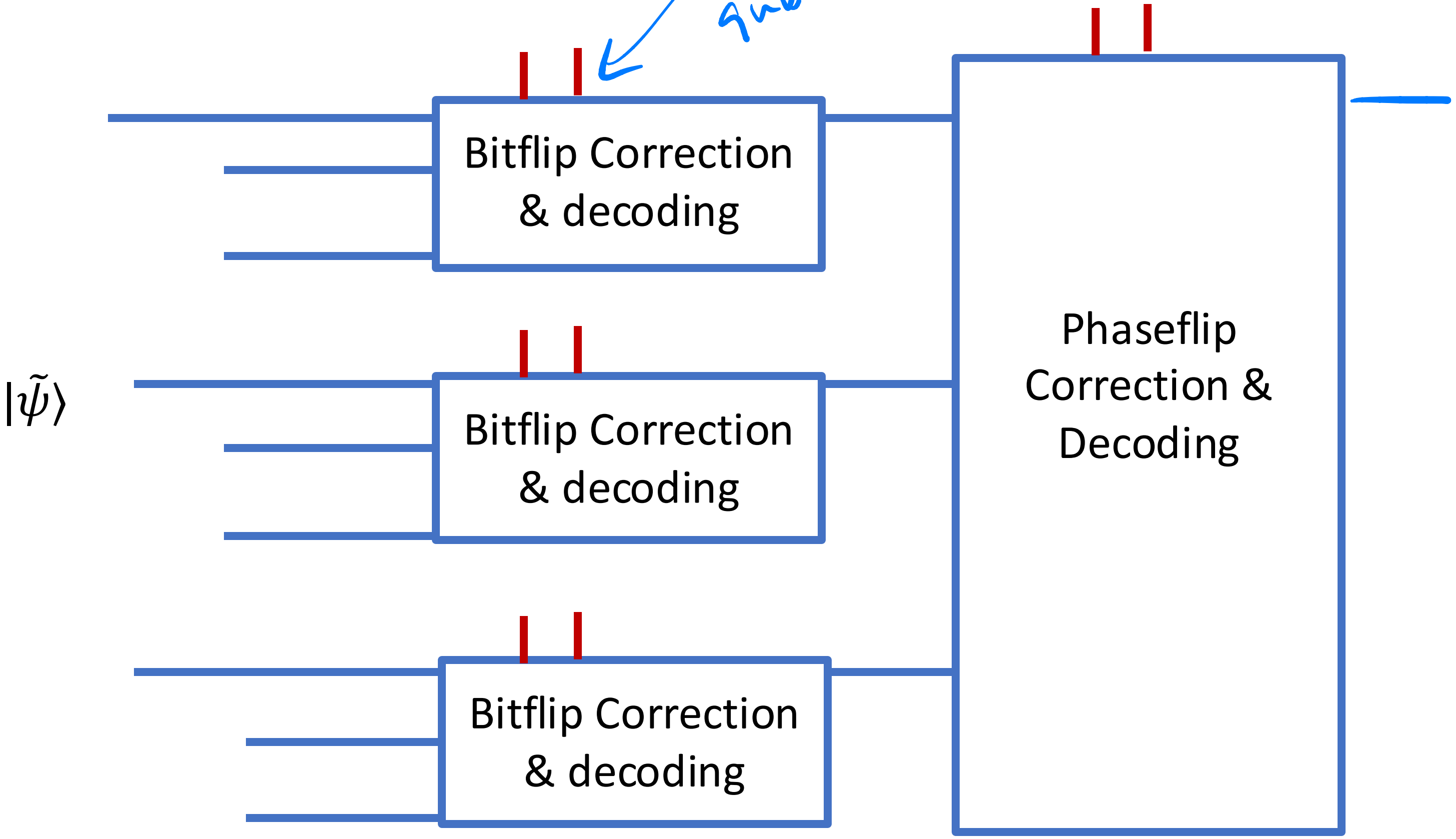
Encoding:



Shor's 9 qubit code

Correction Procedure:

syndrome qubits



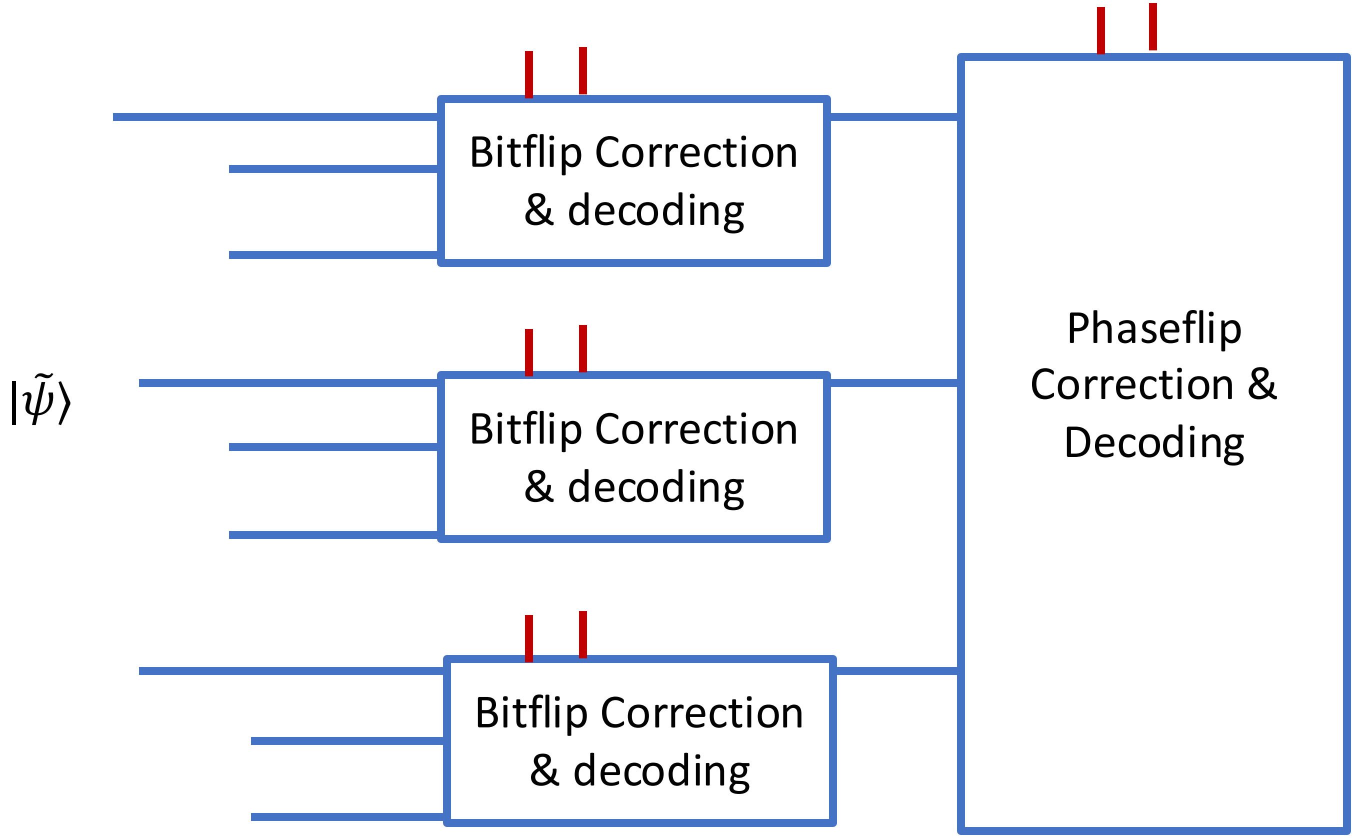
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

(hopefully).

Shor’s 9 qubit code

Case 1: $|\tilde{\psi}\rangle = X_j|\overline{\psi}\rangle$

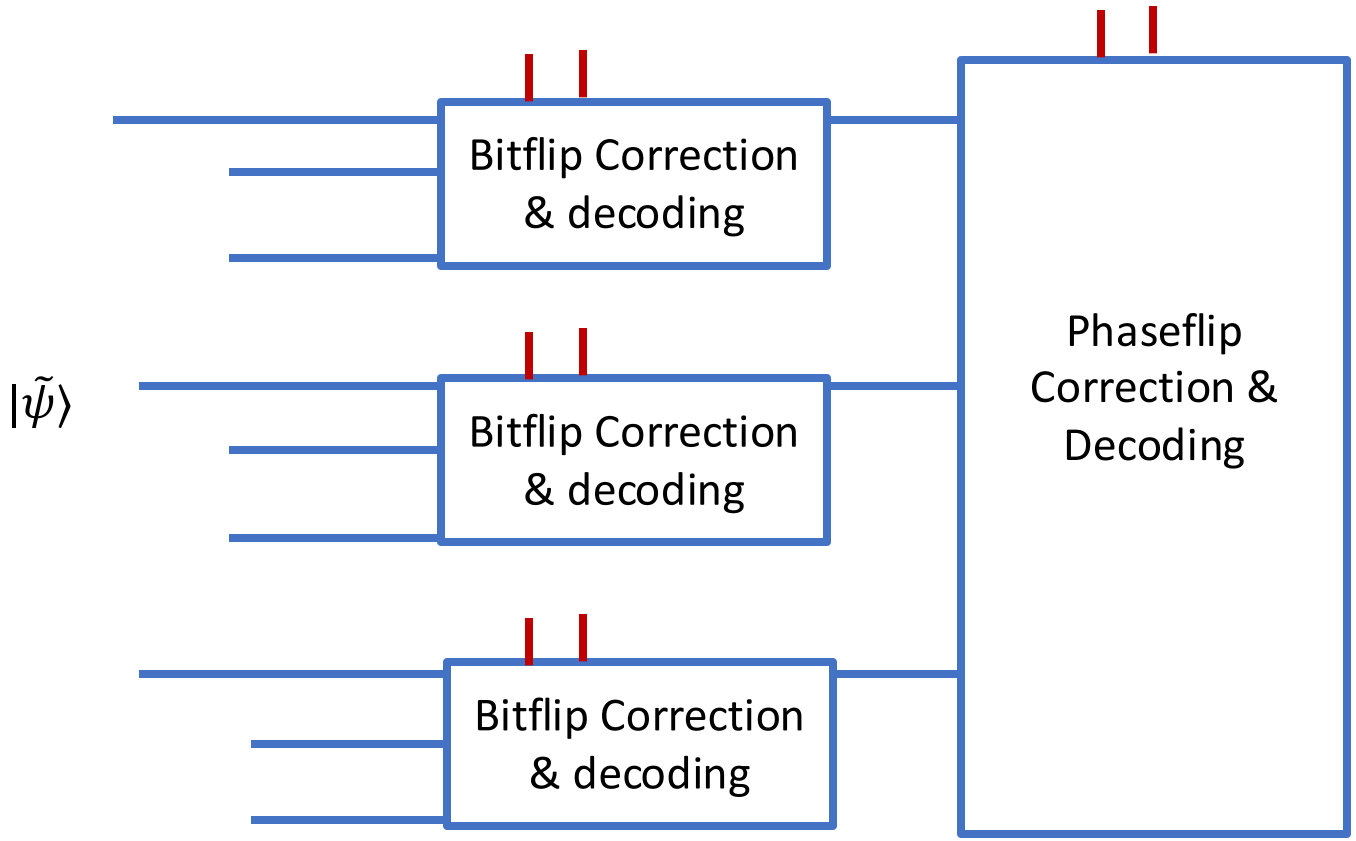
Correction Procedure:



Shor’s 9 qubit code

Case 2: $|\tilde{\psi}\rangle = Z_j|\overline{\psi}\rangle$

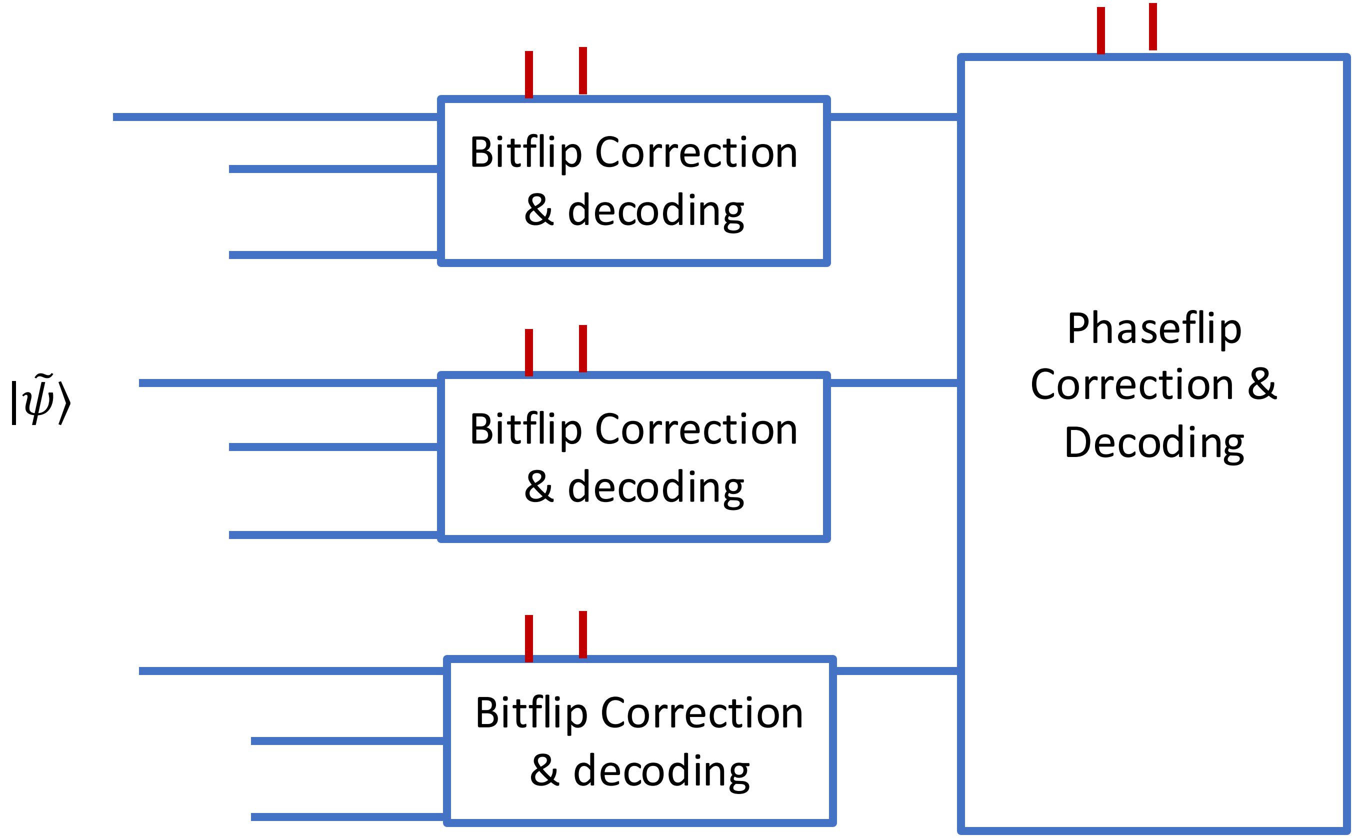
Correction Procedure:



Shor's 9 qubit code

Case 3: $|\tilde{\psi}\rangle = X_j Z_j |\overline{\psi}\rangle$

Correction Procedure:



Why errors can be discretized

- Every unitary can be written as a linear combination of Pauli errors.

Next time

- Glimpse into more sophisticated error correcting codes

- Quantum fault-tolerance