Week 11: Quantum complexity theory

COMS 4281 (Fall 2024)

Admin

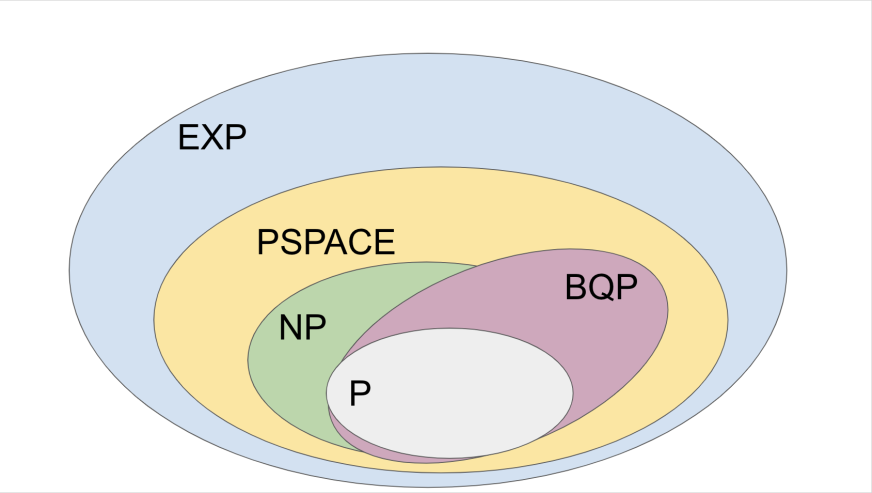
1. Worksheet and Quiz 6 out tonight.
2. Pset2 will be out this weekend, and due December 4.
3. Final in-class exam on December 9.

Last time

Introduction to computational complexity theory. Some basic complexity classes:

- P, BPP - efficiently solvable problems (on classical computers)
- NP - efficiently verifiable problems (on classical computers)
- PSPACE - problems solvable using polynomial memory
- EXP - exponential time-solvable problems (on classical computers)
- BQP - efficiently solvable problems (on quantum computers)

The complexity landscape



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BQP vs NP

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Decision problems where the "yes" inputs have solutions that can be efficiently checked.

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Example: Traveling Salesman Problem

Input: Graph G, integer k

Output: Does G have a tour of length at most k that visits every city once and returns to origin?

TSP is in NP because given a proposed tour, one can check whether it visits every city once, returns to origin, and has length at most k.

NP - Equivalent Definition

A decision problem has an NP "algorithm" if the algorithm can *nondeterministically* guess a solution (if it exists), and check the solution in polynomial time. If there is no solution, no guess will be accepted.

A common misconception

Quantum computers solve problems by trying all possible solutions simultaneously and outputting a correct one!

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If that were true, then NP complete problems would be instantly solvable on quantum computers.

However this simplistic picture of quantum computing is not true. The truth is much more interesting...

Evidence that NP $\not\subseteq$ BQP

We don't have the tools available to outright prove that BQP cannot solve NP-complete problems. But we can try to give evidence for this.

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We will show a **blackbox separation** (also known as a **oracle separation**) between NP and BQP.

Computing with black boxes

Decision problems are defined with inputs that are represented as binary strings (i.e. graphs, integers, etc. are represented in binary), and the outputs are about whether the inputs satisfy some property.

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In particular, cannot "look inside the black box" in order to solve the problem.

Unstructured search problem

Input: A black box function $f : \{0,1\}^n \to \{0,1\}$.

Output: Does there exist x such that $f(x) = 1$?

Unstructured search problem

There is an **NP oracle algorithm** that "solves" the Unstructured Search Problem with 1 query.

If there is a solution x, the NP oracle algorithm guesses x and queries f to see check if $f(x) = 1$.

Unstructured search problem

There is a **quantum oracle algorithm** (namely, Grover's algorithm) that solves the Unstructured Search Problem with $O(\sqrt{2^n})$ quantum queries.

Question: is there a quantum algorithm that can solve Unstructured Search Problem with $\mathrm{poly}(n)$ queries?

Lower bound on unstructured search

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Suppose there was a quantum algorithm A that could find a marked input to $f : \{0,1\}^n \to \{0,1\}$ using $T \ll \sqrt{2^n}$ queries to O_f.

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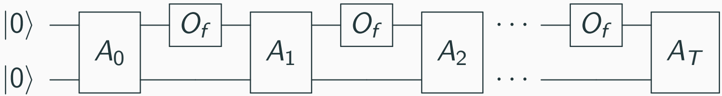
We will show that A can't increase the amplitude on the marked input fast enough to notice it.

Modeling the algorithm

The algorithm A alternates between fixed unitaries $A_0, A_1, \ldots, A_T$ that don't depend on f, and phase oracles O_f.



It starts in the all zeroes state, and either outputs a marked input $|x^*\rangle$ with high probability or outputs "NO MARKED INPUT".

Hybrid method

Imagine running the algorithm A on the all zeroes function (there is no marked input). This is equivalent to running the algorithm with the phase oracle I.

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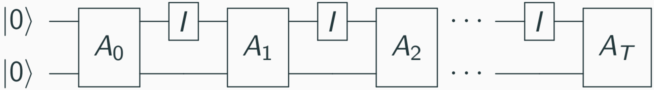

Let $|\psi_T^I\rangle$ denote the final state of the algorithm. Measuring it yields "NO MARKED INPUT" with high probability.

Goal: Show there exists $f(x)$ with unique solution x^*, where running A with O_f yields output state $|\psi_T^f\rangle$ satisfying

$$\Big\| \, |\psi_T^I\rangle - |\psi_T^f\rangle \, \Big\| \leq O(T/\sqrt{2^n}) \ll 1.$$

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Measuring $|\psi_T^f\rangle$ yields "NO MARKED INPUT" with high probability, which means the algorithm **Failed**.

Identifying an “overlooked” input

Define $|\psi_t^I\rangle$ = state of algorithm querying I right after t'th query, for $1 \leq t \leq T$.

$$|\psi_t^I\rangle = \sum_{x,w} \alpha_{t,x,w} \underbrace{|x\rangle}_{\text{query register}} \otimes \underbrace{|w\rangle}_{\text{workspace}}$$

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Intuition: There has to be an x^* where the total amplitude over all timesteps is small.

Identifying the "overlooked" input

Define the **query magnitude** of $x \in \{0,1\}^n$

$$M_x = \sum_{t=1}^{T} \sum_{w} |\alpha_{t,x,w}|^2.$$

Claim: $\sum_x M_x = T$

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$= T$.

Identifying an "overlooked" input

Therefore there exists x^* where $M_{x^*} = T/2^n$.

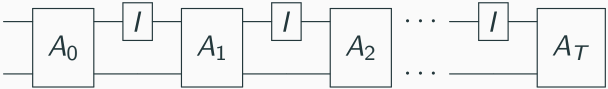
Define $f(x)$ to have x^* as a unique solution.

This x^* is an "overlooked" input - it does not get a lot of attention from the algorithm!

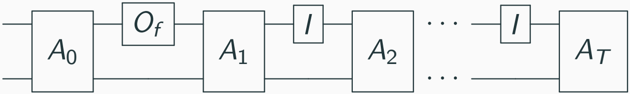
Hybrid method

To show that $|\psi_T^f\rangle$ is close to $|\psi_f^I\rangle$, we analyze **hybrids**, which are fictitious runs of the algorithm where the oracle changes in the middle.

Hybrid $\mathcal{H}_0$:

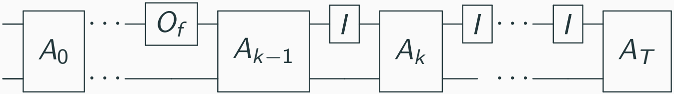


Hybrid $\mathcal{H}_1$:

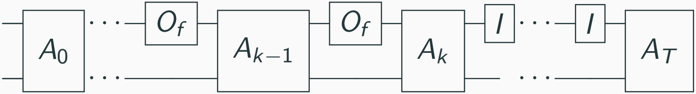


Hybrid method

Hybrid $\mathcal{H}_{k-1}$: first $k-1$ queries to f, last $T-(k-1)$ queries to I.



Hybrid $\mathcal{H}_k$:



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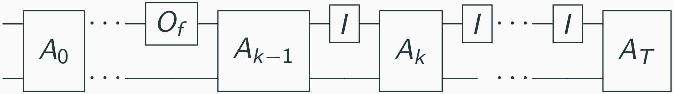

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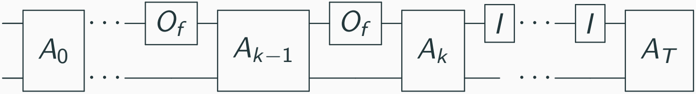

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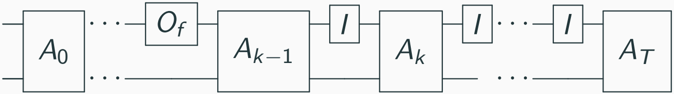

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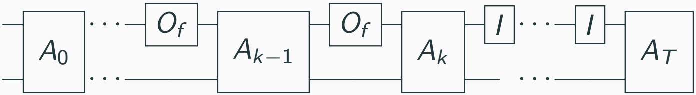

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Observation #1: $|\psi_t^{(t)}\rangle = (O_f \otimes I)\,|\psi_t^{(t-1)}\rangle$

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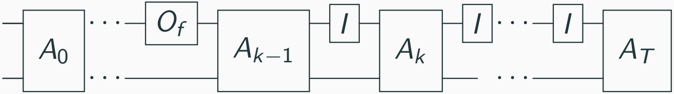

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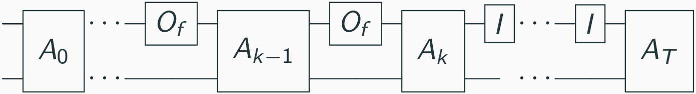

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Observation #2:

$$|\psi_T^{(t-1)}\rangle = A_T \cdot A_{T-1} \cdots A_t \, |\psi_t^{(t-1)}\rangle$$
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Hybrid method

Observation #3: Since applying the **same** unitary to two vectors does not change their distance,

$$\left\| A_T \left|\psi_T^{(0)}\right\rangle - A_T \left|\psi_T^{(T)}\right\rangle \right\| = \left\| \left|\psi_T^{(0)}\right\rangle - \left|\psi_T^{(T)}\right\rangle \right\| .$$

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which by triangle inequality is at most

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Hybrid method

For each t, we can write

$$|\psi_t^{(t-1)}\rangle = \sum_{x,w} \alpha_{t,x,w} |x\rangle \otimes |w\rangle$$

where $|x\rangle$ = query register and $|w\rangle$ = workspace register.

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For each t, we can write

$$|\psi_t^{(t-1)}\rangle = \sum_{x,w} \alpha_{t,x,w} |x\rangle \otimes |w\rangle$$

where $|x\rangle =$ query register and $|w\rangle =$ workspace register.

Then

$$\begin{aligned} \left\| \, |\psi_t^{(t-1)}\rangle - (O_f \otimes I) |\psi_t^{(t-1)}\rangle \, \right\| &= \left\| 2 \sum_w \alpha_{t,x^*,w} |x^*\rangle \otimes |w\rangle \, \right\| \\ &= \sqrt{2 \sum_w |\alpha_{t,x^*,w}|^2} \, . \end{aligned}$$

Hybrid method

Putting everything together:

$$\begin{aligned}
\left\| A_T \left|\psi_T^{(0)}\right\rangle - A_T \left|\psi_T^{(T)}\right\rangle \right\| &\leq \sum_{t=1}^{T} \sqrt{2 \sum_{w} |\alpha_{t,x^*,w}|^2} \\
&\leq \sqrt{2T \cdot \sum_{t=1}^{T} \sum_{w} |\alpha_{t,x^*,w}|^2} \\
&= \sqrt{2TM_{x^*}} \\
&\leq \sqrt{2T^2/2^n} = O(T/\sqrt{2^n}).
\end{aligned}$$

Quantum advantage

We're still far away from being able to run Grover's algorithm or Shor's factoring algorithm. How to demonstrate quantum advantage in the near term?

Quantum supremacy task

We wish to find a computational task T that:

1. NISQ (Noisy Intermediate-Scale Quantum) machine can run T in, e.g. < 1 second.
2. Verifiable on a classical computer in a reasonable amount of time (e.g. several weeks on a supercomputing cluster)
3. Some complexity evidence that T cannot be efficiently solved by classical computers.

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3. Some complexity evidence that T cannot be efficiently solved by classical computers.

Such a task T would demonstrate the supremacy of quantum computers over classical computers.

Quantum supremacy task

This computational task T can only happen in a "sweet spot" with $\sim$ 50-100 qubits.

Enough that it's not easy for classical computers, but not too much so that we can run it on our existing quantum computers, and also verify it using $\sim 2^{50}$ operations on a classical computer.

Random circuit sampling

Number of qubits $n = 50$
Number of gates $m = 200$
Number of samples $T =$ several million

1. Pick a random quantum circuit C acting on n qubits, with m gates.
2. Using quantum computer run circuit C on $|0 \cdots 0\rangle$, generate samples $x_1, \ldots, x_T$ from the distribution $\mathcal{D}_C$.
3. Output $x_1, \ldots, x_T$.

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The distribution $\mathcal{D}_C$: each sample x occurs with probability $p_C(x) = |\langle x| C |0 \cdots 0\rangle|^2$.

Hardness of random circuit sampling?

A quantum computer, by definition, can easily perform the random circuit sampling task. How hard is it for a classical computer to do the same?

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A quantum computer, by definition, can easily perform the random circuit sampling task. How hard is it for a classical computer to do the same?

Theorem (Bouland, Fefferman, Nirkhe, Vazirani 2019): There is no classical algorithm that, given circuit C, can generate samples from $\mathcal{D}_C$ with high probability, unless the polynomial hierarchy collapses to the third level.

Hardness of random circuit sampling?

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This theorem talks about sampling *exactly* from $\mathcal{D}_C$. However, not even the quantum computer can do that, because it has some small amount of noise. One can ask how hard is it to *approximately sample* from $\mathcal{D}_C$. It is conjectured that this is still hard for classical algorithms (assuming polynomial hierarchy does not collapse).

Verification of random circuit sampling

How do you check whether a bunch of samples $x_1, \ldots, x_T$ were generated by $\mathcal{D}_C$? There is no known efficient way of doing so.

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Idea: Heavy output generation (HOG) test

1. Use a classical supercomputer to compute the median α of all $p_C(x)$.
2. If at least $2/3$ of $x_1, \ldots, x_T$ are **heavy** (meaning $p_C(x_i) \geq \alpha$), then accept. Otherwise reject.

Verification of random circuit sampling

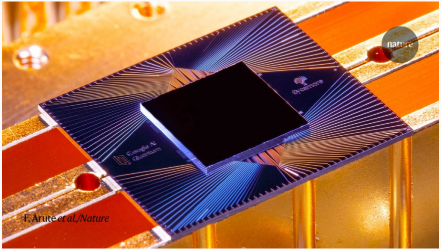
Intuition: heavy x's form the bulk of the probability mass of distribution $\mathcal{D}_C$. Sampling from $\mathcal{D}_C$ should yield a lot of heavy strings.

However, it should also be hard for a classical computer to predict, given a circuit C and x, whether $p_C(x)$ is above the median.

If a quantum machine is able to consistently output heavy strings from $\mathcal{D}_C$, then the machine must've "done something right".

Experimental demonstrations

Conducted in Fall 2019. Ran many circuits on their 49-qubit "Sycamore" chip.



Extremely noisy: signal-to-noise ratio is about 1%. (However, Google claims it is enough to verify the sampling).

Recently: many challenges to this claim (faster classical simulations, noisy sampling may not be as hard as we thought).

Next time

Mixed states.