Area Laws in One Dimension

Juan Castaneda

December 2020

Abstract

It is well known that the ground states of local Hamiltonians can exhibit multi-partite entanglement. The entanglement of these ground states is important in many-body physics; for example, it can lead to peculiar forms of matter such as Bose-Einstein condensates. Area laws can be used to bound the entanglement complexity of ground states of certain types of 1-D Hamiltonians by a constant. This report reviews two seminal papers that opened the doors for computer scientists to investigate area laws and provide better bounds for the entanglement of ground states of 1-D Hamiltonians. Some extensions of 1-D area laws are also discussed, including how area laws lead to the possibility of approximating such ground states efficiently, and in such a manner that the representation of the approximate ground state is both useful for the simulation of quantum many-body systems and provides information about the structure of entanglement present in the state.

Contents

1 Introduction

Area laws provide bounds on the complexity of entanglement that ground states of a quantum system can exhibit. Consider a system where particles are placed on a lattice. Area laws say that the entanglement entropy between two partitions of a quantum state of such a system can, at most, grow proportionally with the boundary between the two partitions, as demonstrated in Figure 1.1. In one dimension (ie. a 1-D chain), this means that the entanglement entropy of the ground state across any cut must be constant with respect to the number of particles.

One of the main motivations behind deriving area laws comes down to knowing whether or not we are able to come up with efficient classical representations of the ground states of a Hamiltonian. If the entanglement of the ground state is highly complex, this task becomes intractable. However, area laws allow us to put an upper bound on the complexity of entanglement, and as a result we can come up with efficient approximations of the ground states using tools such as matrix product states, which are useful for simulating quantum many-body systems as well as understanding the structure of the entanglement of quantum states.

This paper will discuss area laws for one dimensional systems, including how specific bounds on the ground state entanglement were derived, as well as the importance of not only the area laws themselves, but their proofs as well. In addition, some of the applications and extensions of area laws will be touched upon.

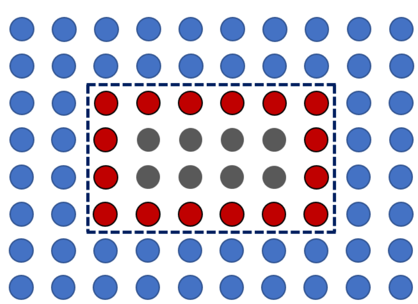


Figure 1.1: Particles on a lattice separated into two sets: those within the dashed line, and those outside. According to the area law, the entanglement entropy of the ground state between these two partitions depends only on the boundary (the red particles).

2 Background Concepts

2.1 Local, Gapped, and Frustration-Free Hamiltonians

Definition 2.1 (k-Local Hamiltonian)**.** *A Hamiltonian is k-local if it can be written as the sum of Hamiltonians (referred to as **Hamiltonian terms**) that act on at most k qubits.*

Definition 2.2 (Gapped Hamiltonian)**.** *In the context of 1-D area laws, this refers to constant-gapped Hamiltonians. These are Hamiltonians for which the ground state energy is zero, and the next lowest eigenvalue is greater than or equal to some constant ϵ, referred to as the **spectral gap**.*

Definition 2.3 (Frustration-Free Hamiltonian)**.** *If the ground states of the total Hamiltonian are also ground states of each of the local Hamiltonian terms, then the Hamiltonian is considered to be frustration-free. Otherwise, it is referred to as frustrated.*

2.2 Entanglement Complexity

2.2.1 Schmidt Rank

Theorem 2.4 (Schmidt Decomposition)**.** *Consider a pure state $|\psi\rangle$ whose qubits are partitioned into two non-intersecting sets A and B (ie. $|\psi\rangle$ is a vector on the tensor product of the corresponding Hilbert spaces: $\mathcal{H}_A \otimes \mathcal{H}_B$). Then there exist orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ such that*

$$|\psi\rangle = \sum_i \alpha_i |i\rangle_A |i\rangle_B$$

*This is called the **Schmidt decomposition** of $|\psi\rangle$.*

The coefficients α_i are called the **Schmidt coefficients** and are non-negative real numbers which satisfy $\sum_i \alpha_i^2 = 1$.

The **Schmidt rank** of $|\psi\rangle$ is the number of non-zero Schmidt coefficients in its decomposition.

2.2.2 Entanglement Entropy

Entanglement entropy is a measure of the degree of entanglement of a bipartite state. With the Schmidt decomposition of such a state in mind, the Von Neumann **entanglement entropy** of $|\psi\rangle$ is defined as

$$S = -\sum_i |\alpha_i|^2 log(|\alpha_i|^2)$$

3 1-D Area Laws

In [9], Hastings first showed that the entanglement entropy across any cut in a gapped 1-D system was bounded by a constant that was independent of the number of particles in the system. These results implied that the problem of approximating ground states of such Hamiltonians is in the **NP** complexity class [6]. However, the proof of this involved complicated analytical methods. Aharonov et al. [2] followed up with a proof for a specific, yet rich, class of Hamiltonians that made the subject of area laws more accessible to computer scientists. This simplified proof relies on the Detectability Lemma, which was introduced by the same authors in a previous paper [3], but in [2] it is presented in a more basic context than the previous work.

3.1 Detectability Lemma

The **Detectability Lemma (DL)** in [2] considers a 1-D gapped, frustration-free, local Hamiltonian whose Hamiltonian terms are positive semi-definite and have a norm of at most 1. In particular, the system under consideration is one where particles are placed on a line and only exhibit nearest-neighbor interactions (thus, the Hamiltonian terms are 2-local). The goal of the **DL** is to show that the ground state of such a system can be approximated by applying a local operator to an unentangled product state. This local operator must therefore approximate the ground state projection operator.

Let P_i be the projection onto the ground state of the Hamiltonian term H_i. Since each H_i is 2-local, the terms can be partitioned into two sets - one containing odd numbered terms, and another containing even number terms - such that the terms in each set do not act on any of the same particles. This is demonstrated in Figure 3.2. Two operators are then defined as the products of the projections onto the terms in each set. That is:

$$\prod_{odd} = P_1 P_3 P_5 ... \qquad \prod_{even} = P_2 P_4 P_6 ...$$

The operator that can be used to approximate projection onto the ground space is then

$$A = \prod_{odd} \prod_{even}$$

In simple terms, the **DL** says that applying the local operator A to a state reduces the overlap between that state and the orthogonal complement of the ground space by a constant factor. It can be thought of as "getting rid" of parts of the state not in the ground space of the Hamiltonian. More precisely:

Lemma 3.1 (Detectability Lemma)**.** *Let $A = \prod_{odd} \prod_{even}$, and $\mathcal{H}'$ be the orthogonal complement of the ground space of the Hamiltonian. Then*

$$\|A|_{\mathcal{H}'}\| \leq \frac{1}{(\epsilon/2+1)^{1/3}}$$

where ϵ is the spectral gap of the Hamiltonian [2]

This means that applying the operator A multiple times gives a better and better approximation of the projection onto the ground state. In particular, applying this operator l times approximates the projection onto the ground state exponentially well:

$$A^l = P_{\text{ground state}} + e^{-\mathcal{O}(l)}$$

where $P_{\text{ground state}}$ is the projection onto the ground state.

3.2 1-D Area Law

3.2.1 Statement of the Area Law

Continuing with the 1-D system we have considered in section 3.1, the area law in 1-D provides an upper bound to the entanglement entropy (as defined in section **2.2.2**) across any cut along the chain.

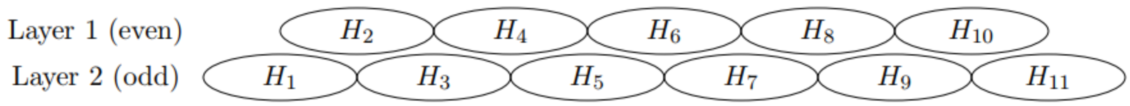


Figure 3.2: The even and odd 2-local Hamiltonian terms partitioned into two layers. No two terms in the same layer act on the same particle. [2]

Although the particles are set up along one dimension, each particle is allowed to have a dimension $d \geq 2$. For example, $d = 2$ for a system of qubits. The upper bound given by the 1-D area law depends on this dimensionality, as well as on the spetral gap of the Hamiltonian. We now consider a cut along the chain of particles that splits the particles into two connected chains, A and B, on either side of the cut.

Theorem 3.2 (Area Law for 1-D Frustration-Free Gapped Hamiltonians)**.** *For any cut along the chain of particles, the magnitude of the entanglement entropy, S, of the ground state across the cut satisfies the following inequality:*

$$S \leq \frac{10}{\delta} d^{4/\delta} (ln(d))^2$$

where d is the dimensionality of each particle, and δ is defined in terms of the spectral gap, ϵ, as $\delta = 1 - (1 + \epsilon/2)^{-1/3}$ [2]

This means that if the system is in the ground state of the Hamiltonian, the particles on one side of the cut cannot share infinitely complex entanglement with the particles on the other side of the cut; the entanglement entropy must be finite in this case. It is significant to note that this bound on the entanglement entropy is constant with respect to the number of particles in the system.

3.2.2 Overview of Proof

The proof of this 1-D area law is the starting point from which much future work in the subject of area laws built upon. Thus it is valuable to understand the proof, as its relative simplicity demonstrates how it opened the doors for computer scientists to research area laws.

The proof of the area law in Theorem 3.2 relies on two key lemmas, both of which can be proved using the **DL**.

Lemma 3.3. *For any cut along the chain of particles, there exists a product state $|\psi_1\rangle_A \otimes |\psi_2\rangle_B$ which has a constant, non-zero, overlap with the ground state:*

$$\| \langle \Omega | (|\psi_1\rangle_A |\psi_2\rangle_B) \| \geq \mu$$

where $|\Omega\rangle$ is the ground state, $|\psi_1\rangle_A$ describes the state of the particles on one side of the cut (and similarly $|\psi_2\rangle_B$ for those on the other side), and μ is a non-zero constant that depends on the spectral gap of the Hamiltonian and the dimensionality of the particles.[2]

The following is a summary of the proof of Lemma 3.3:

1. Let ρ^{2l} be the density matrix of the ground state Ω, limited to the l particles on either side of a cut, for some large enough constant l. Let ρ^l_A and ρ^l_B be the reduced density matrices of

Ω restricted to the l particles on the left and right side of the cut respectively. Thus, $\rho_A^l \otimes \rho_B^l$ is referred to as the "disentangled" version of ρ^{2l}.

2. By contradiction to Lemma 3.3, assume that the overlap between Ω and any product state $|\psi_1\rangle_A \otimes |\psi_2\rangle_B$ is smaller than some constant, μ. This includes the product state associated with $\rho_A \otimes \rho_B$ (notice this is not limited to the l particles on either side of the cut). Then there must exist a measurement (the projection onto the ground state) that distinguishes between ρ^{2l} and $\rho_A^l \otimes \rho_B^l$ with a constant, non-zero, probability. The **DL** from section 3.1 can be used to approximate this measurement by applying the operator A $l/2$ times [2]

3. The existence of such a measurement indicates that the entanglement across the cut is significantly large, as that is the difference between the two states which caused them to be distinguishable. Slightly more formally,

$$S(\rho_A^l) + S(\rho_B^l) - S(\rho^{2l}) \leq \frac{\delta}{2}l - 1$$

which just says that the difference between the entanglement entropy of ρ^{2l} and the sum of the entanglement entropies of ρ_A^l and ρ_B^l must be at least some constant that depends on δ and l. Note that the value of l is essentially a measure of how good the ground state approximation is, since the higher the value of l, the more times the operator A from the **DL** is applied (recall that A moves a state closer to the ground state each time it is applied).

4. Finally, it is shown that applying this inequality recursively (to cuts located at the halfway point of each of the two segments made the previous cut) leads to a contradiction in the value of the entanglement entropies of ρ_A^l and ρ_B^l. This implies that the assumption in step 2 must be false, proving the result of Lemma 3.3: the overlap between the ground state and the product state associated with $\rho_A \otimes \rho_B$ is at least μ.

From the proof of Lemma 3.3, the value of μ is ultimately shown to be:

$$\mu = d^{-l}(1-\delta)^{l_0/4}$$

where $l_0 = d^{4/\delta}$ [2]

The second lemma starts by assuming the result of the first:

Lemma 3.4. *If there exists a product state $|\psi_1\rangle_A \otimes |\psi_2\rangle_B$ such that $\|\langle\Omega|(|\psi_1\rangle_A|\psi_2\rangle_B)\| \geq \mu$, then the entanglement entropy of the ground state across the cut is bounded by*

$$S \leq \frac{3}{\delta}\left(ln(\frac{1}{\mu^2\delta}) + 2\right) ln(d)$$

[2]

To understand the reasoning behind the proof of this Lemma, first recall that the A operator from the **DL** lets us get the state $|\psi_1\rangle_A \otimes |\psi_2\rangle_B$ exponentially close to the ground state without significantly increasing the Schmidt rank. Aharonov et al. [2] show that after applying A l times, the resulting state is close to the ground state and has a constant upper bound on its Schmidt rank.

Now, consider the following fact:

Fact 3.5. *Given a state vector* $|\Omega\rangle$ *on* $\mathcal{H}_A \otimes \mathcal{H}_B$*, the largest inner product between* $|\Omega\rangle$ *and any normalized vector* $|\psi\rangle$ *with Schmidt rank* r *is given by*

$$\langle\phi||\psi\rangle \leq \sqrt{\sum_{i=1}^{r} \lambda_i}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots$ *are the eigenvalues of the reduced density matrix of* $|\psi\rangle$ *[2]*

Using this fact, we see that the bound (r) on the Schmidt rank of the approximate ground state provides an upper bound on the sum of the d^{2l} largest Schmidt coefficients of the ground state itself (the norm squared of the i^{th} Schmidt coefficient is equal to λ_i).

Finally, this bound can be used to show that the worst case upper bound on the entanglement entropy of the ground state across any cut is that given in Lemma 3.4

Since $d \geq 2$ and l is a large integer, we get that $\mu \leq \delta$. It can also be shown that $ln(\frac{1}{\mu}) \leq 6$ by making some assumptions about the size of the spectral gap. From there, the two lemmas can be applied in succession to arrive at the area law of Theorem 3.2.

3.3 Exponential Improvement of the 1-D Area Law

In a follow-up paper to [2], Arad et. al. presented a new 1-D area law for gapped Hamiltonians that greatly reduced the upper bound on the entanglement entropy. [6]

3.3.1 Improved 1-D Area Law

The improved 1-D area law in [6] states that the entanglement entropy across any cut in the chain of particles is bounded by

$$S \leq \mathcal{O}(\frac{log^3 d}{\epsilon})$$

This is clearly a tighter bound than that of [2], and in fact is exponentially better than Hastings' original bound [9], which was $e^{\mathcal{O}(\frac{log(d)}{\epsilon})}$ [6]. This is important because the proof for this improved area law, similar to Hastings' result, doesn't depend on the Detectability Lemma. This means that unlike the area law in [2], it is not limited to frustration-free Hamiltonians. Instead, this new area law can be applied to frustrated Hamiltonians as well.

3.3.2 Overview of Proof

Similar to the **DL** strategy, the key is to find an operator that can be applied to a product state repeatedly in order to approximate the ground state of this new Hamiltonian, without making the entanglement rank too large.

Such an operator is referred to as an **Approximate Ground State Projector (AGSP)**, which is loosely defined as follows:

Definition 3.6 (Approximate Ground State Projector (AGSP))**.** *An operator,* K*, is a* (D, Δ)*-AGSP if the following properties hold*

1. *Applying* K *to any quantum state increases its entanglement rank by at most a factor of* D

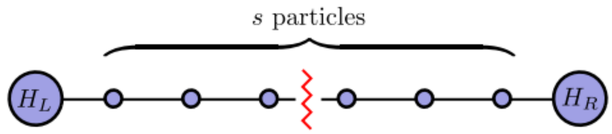


Figure 3.3: The modified Hamiltonian, $H^{(t)}$. H_L (H_R) is the sum of all the terms to the left (right) of the cut that were truncated. The particles in between H_L and H_R adhere to their original Hamiltonian terms [6]

2. *For any quantum state, $|\psi^{\perp}\rangle$, in the orthogonal complement of the ground space of the Hamiltonian:*
   $\left\| K|\psi^{\perp}\rangle \right\|^2 \leq \Delta$

As was the case with the **DL**, the second property is to ensure that applying the operator to a state moves it closer to the ground state by decreasing its overlap with the orthonormal complement of the ground space.

With the two properties of an AGSP above, by arguments similar to the ones made in section 3.2.2, it can be shown that the following theorem is true:

Theorem 3.7 (Area Law)**.** *If there exists a (D, Δ)-AGSP such that $D \cdot \Delta \leq \frac{1}{2}$, then the entanglement entropy of the ground state is bounded by*

$$S \leq \mathcal{O} \cdot log(D)$$

[6]

All that is left is to construct a suitable AGSP. Begin by truncating the Hamiltonian terms that act on particles outside of the s+1 particle neighbourhood of the cut (ie. far from the cut). The truncation involves replacing any eigenvalues of the terms that are larger than some t with t itself. The resulting Hamiltonian, $H^{(t)}$, is shown in Figure 3.3. This modification allows the norm of the Hamiltonian to be bounded by $u = s + 2t$. [6]

In the frustration-free case, the ground state of this new Hamiltonian is the same as that of the 1-D chain, but if the Hamiltonian is frustrated there must be a compromise between the norm of the new Hamiltonian and how close its ground state is to that of the original Hamiltonian.

The AGSP, K, is chosen to be a Chebyshev Polynomial of degree l ($C_l(x)$) whose argument is the truncated Hamiltonian (more explanation on these polynomials and how they are used to construct such operators can be found in [6], as well as in a previous paper by Arad et. al. [5]). This choice of $K = C_l(H^{(t)})$ implies that $\Delta = e^{-\Omega(l\sqrt{\epsilon/u})}$, where Ω refers to "Big Omega" notation. Furthermore, it can be shown that the entanglement rank of K is at most $D = (dl)^{\mathcal{O}(l/s+s)}$. [6]

For the frustration-free case, if l and s are chosen to be $l = \mathcal{O}(s^2)$ and $s = \tilde{\mathcal{O}}(\frac{log^2 d}{\epsilon})$, Theorem 3.7 can then be applied to get the area law in section 3.3.1. For the frustrated case, Arad et al. [6] first demonstrate that the ground states of $H^{(t)}$ are exponentially close in t to the ground states of H and that the spectral gaps of the two Hamiltonians are of the same order. Then, rather than apply the same AGSP several times to approximate a projection onto the ground state, for a carefully chosen sequence of values of t, the associated AGSPs are applied to the desired product state. This is done in lieu of direct application of Theorem 3.7 in order to guide a state towards the ground space of H without increasing the entanglement rank too much, which is what would happen should Theorem 3.7 be applied directly. This process results in the area law of section 3.3.1, but now demonstrates that the area law holds for frustrated Hamiltonians as well.

4 Efficient Ground State Approximation

The 1-D area laws that have been described demonstrate that ground states of 1-D gapped Hamiltonians can be approximated relatively efficiently. As previously stated, Hastings' area law implied that the task of approximating these ground states is in the **NP** complexity class. Furthermore, Arad et. al. [6] show that there exists an algorithm for approximating these ground states that is sub-exponential, which provides evidence that this task is not **NP-Hard**.

4.1 Tensor Networks

Just like specifying the coefficients of a wave function, tensor networks are a way of representing a quantum state classically. Tensor networks provide several advantages. Rather than writing out a wavefunction, which can be extremely complicated for highly entangled states, tensor networks are represented by drawing tensor network diagrams, which capture the entanglement properties of a system. This is one benefit over simply specifying coefficients of a wavefunction which doesn't make apparent the structure of the entanglement between different components of the system.

Tensor networks are especially useful for efficiently representing quantum states. A general quantum state of n qubits can be written as

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_n} C_{i_1 i_2 \ldots i_n} |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$$

for $i = 1, 2$ (since qubits have 2 degrees of freedom).[12]

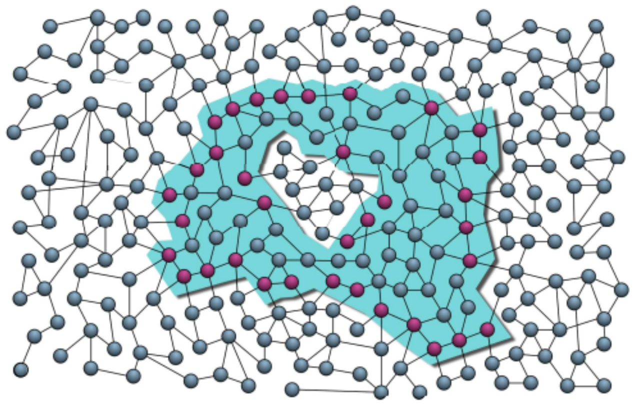
There are 2^n numbers $C_{i_1 i_2 \ldots i_n}$, which means that the number of parameters required to describe the system is exponential in n. These numbers can be thought of as coefficients that make up a tensor, C, of rank n. By replacing this large tensor with a tensor network of smaller-rank tensors, the state can be represented more efficiently. [12]

The tensors in the resulting tensor network are connected by indices that represent the entanglement between different parts of the many-body system. The number of different values these indices can take is referred to as bond dimensions, and the higher their value, the more quantum correlations are present in the state. The largest of these is called the **bond dimension** of the tensor network. [12]

4.2 Matrix Product States

Matrix product states (**MPS**) are a a type of tensor network state that corresponds to a 1-D array of tensors, and are the basis of some very powerful algorithms for simulating 1-D quantum many-body systems. [12]

MPS can be used to represent any quantum state of an n-particle system, but to cover all possible states the bond dimension must be exponentially large in n. It is known that for 1-D systems, low energy states can be approximated by a **MPS** of bond size that grows polynomial in n. [12]



[7]

Figure 5.4: The subset S lies in the light-blue shaded region. According to a generalized area law, the entanglement entropy of the ground state would depend linearly on the number of edges leaving this shaded area.

Using the results and proof techniques outlined in section 3.3, Arad et al. [6] prove that the ground state of a 1-D gapped frustration-free Hamiltonian can be approximated to within $\frac{1}{poly(n)}$ by a **MPS** with sub-linear bond dimension of size $e^{\mathcal{O}(\epsilon^{-1/3} log^{\frac{3}{2}} n)}$, thus providing proof of a very efficient and useful way of approximating the ground states of such systems.

5 Further Advancements and Open Questions

One logical extension to area laws is what is known as a **generalized area law**. These area laws are not restricted to lattice systems. Instead, they apply to systems whose Hamiltonian terms correspond to edges of an arbitrary graph, as shown in Figure 5.4. In this case, the area law says that for any subset of particles, S, the entanglement entropy between S and $\overline{S}$ is linearly proportional to the number of edges leaving S. [1]

Aharonov et al. disproved this generalized area law in [1] by presenting systems that act as counter-examples. However, in [4], Anshu et al. demonstrate that gapped local Hamiltonian systems, including those that violate the generalized area law for entanglement entropy, still exhibit generalized area law behaviour in the entanglement *spread* of the ground state (which roughly measures the log of the ratio between the largest and smallest Schmidt coefficients of the ground state across a bipartition). This result is not restricted to 1-D systems, and although it applies to any arbitrary interaction graph, the bound on the entanglement spread is improved if the Hamiltonian corresponds to a lattice system. [4] However, entanglement *entropy* area laws in 2-D and higher dimensions remain an open area of research. Though no such area laws have been proven, simulations of 3-D quantum fluids have demonstrated the existence of area law behaviour in higher dimensions. [10]

Another related area of interest is the search for area laws for other classes of Hamiltonians (in 1-D and higher) that exhibit area law behaviour, such as for long-range interacting systems. [8], [11]

6 Conclusion

We have presented area laws for 1-D gapped Hamiltonians, and from their proofs laid out some of the basic tools used in modern research of entanglement area laws, including in higher dimensions (in particular Approximate Ground State Projectors, which were inspired by the Detectability Lemma from [2]). Furthermore, we discussed how these area laws make possible the effective representation of the ground states of these Hamiltonians as matrix product states, allowing for efficient numerical simulation of the ground states. Finally, while proofs of higher dimensional area laws remain elusive, it has been shown that quantum many-body systems different from the class of Hamiltonians first considered by Hastings [9] can exhibit area law behaviour, providing motivation for the continued search.

References

[1] Dorit Aharonov et al. "Local Tests of Global Entanglement and a Counterexample to the Generalized Area Law". In: *2014 IEEE 55th Annual Symposium on Foundations of Computer Science* (Oct. 2014). DOI: 10.1109/focs.2014.34. URL: http://dx.doi.org/10.1109/FOCS.2014.34.

[2] Dorit Aharonov et al. *Quantum Hamiltonian complexity and the detectability lemma*. 2011. arXiv: 1011.3445 [quant-ph].

[3] Dorit Aharonov et al. *The Detectability Lemma and Quantum Gap Amplification*. 2008. arXiv: 0811.3412 [quant-ph].

[4] Anurag Anshu, Aram W. Harrow, and Mehdi Soleimanifar. *From communication complexity to an entanglement spread area law in the ground state of gapped local Hamiltonians*. 2020. arXiv: 2004.15009 [quant-ph].

[5] Itai Arad, Zeph Landau, and Umesh Vazirani. "Improved one-dimensional area law for frustration-free systems". In: *Physical Review B* 85.19 (May 2012). ISSN: 1550-235X. DOI: 10.1103/physrevb.85.195145. URL: http://dx.doi.org/10.1103/PhysRevB.85.195145.

[6] Itai Arad et al. *An area law and sub-exponential algorithm for 1D systems*. 2013. arXiv: 1301.1162 [quant-ph].

[7] J. Eisert, M. Cramer, and M. B. Plenio. "Colloquium: Area laws for the entanglement entropy". In: *Reviews of Modern Physics* 82.1 (Feb. 2010), pp. 277–306. ISSN: 1539-0756. DOI: 10.1103/revmodphys.82.277. URL: http://dx.doi.org/10.1103/RevModPhys.82.277.

[8] Zhe-Xuan Gong et al. "Entanglement Area Laws for Long-Range Interacting Systems". In: *Physical Review Letters* 119.5 (July 2017). ISSN: 1079-7114. DOI: 10.1103/physrevlett.119.050501. URL: http://dx.doi.org/10.1103/PhysRevLett.119.050501.

[9] M B Hastings. "An area law for one-dimensional quantum systems". In: *Journal of Statistical Mechanics: Theory and Experiment* 2007.08 (Aug. 2007), P08024–P08024. DOI: 10.1088/1742-5468/2007/08/p08024. URL: https://doi.org/10.1088%2F1742-5468%2F2007%2F08%2Fp08024.

[10] C. M. Herdman et al. "Entanglement area law in superfluid 4He". In: *Nature Physics* 13.6 (Mar. 2017), pp. 556–558. ISSN: 1745-2481. DOI: 10.1038/nphys4075. URL: http://dx.doi.org/10.1038/nphys4075.

[11] Tomotaka Kuwahara and Keiji Saito. “Area law of noncritical ground states in 1D long-range interacting systems”. In: *Nature Communications* 11.1 (Sept. 2020). ISSN: 2041-1723. DOI: 10.1038/s41467-020-18055-x. URL: http://dx.doi.org/10.1038/s41467-020-18055-x.

[12] Román Orús. “A practical introduction to tensor networks: Matrix product states and projected entangled pair states”. In: *Annals of Physics* 349 (Oct. 2014), pp. 117–158. ISSN: 0003-4916. DOI: 10.1016/j.aop.2014.06.013. URL: http://dx.doi.org/10.1016/j.aop.2014.06.013.