```
In [ ]: %run utils.py
```

COMS 4281 - Intro to Quantum Computing

Problem Set 2, Quantum Info Basics

Due: October 11, 11:59pm

Collaboration is allowed and encouraged (teams of at most 3). Please read the syllabus carefully for the guidlines regarding collaboration. In particular, everyone must write their own solutions in their own words.

Write your collaborators here:

Problem 1: Non-standard Basis Measurements

a) Consider an orthonormal basis $B = \{|b_1\rangle, \ldots, |b_d\rangle\}$ for $\mathbb{C}^d$. As we learned in class, measuring a quantum state $|\psi\rangle \in \mathbb{C}^d$ according to the basis B yields outcome $|b_j\rangle$ with probability $|\langle b_j|\psi\rangle|^2$.

In class we also learned that this process was equivalent to first applying a unitary U on $|\psi\rangle$, and then measuring the resulting state in the standard basis. In other words, the probability of obtaining standard basis outcome $|j\rangle$ when measuring $U|\psi\rangle$ in the standard basis, equal to $|\langle b_j|\psi\rangle|^2$. What unitary U accomplishes this? Give an algebraic expression for U, such as a sum of outer products, or a description of the rows/columns of U, etc. Then prove that it works.

Your Solution:

write your solution here, using LaTeX and Markdown

b) Now let's implement the unitary for measuring in the following basis B:

$$|\psi_0\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$$

and

$$|\psi_1\rangle = -\sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle$$

First, write down the measurement probabilities if we measure the following states in the basis B:

$$|1\rangle, |-\rangle, |+\rangle, \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$$

Your Solution:

write your solution here, using LaTeX and Markdown

c) In the code below, write the matrix U that implements the change of basis from the standard basis to the basis above.

```
In [ ]:
# ========= BEGIN CODE =================

# ========= END CODE =================

def perform_basis_measurement(initial_state: List[float]) -> QuantumCircuit:
    qr = QuantumRegister(1, name="input state")
    cr = ClassicalRegister(1, name="output")
    qc = QuantumCircuit(qr, cr)
    qc.initialize(initial_state)
    qc.append(UnitaryGate(U), qr)
    qc.measure(qr, cr)
    return qc
```

Now we'll test your basis change on some states and plot their measurement statistics. You should use this to check whether you implemented the right basis change U.

```
In [ ]:
#First, we test it on the |1> state
qc1 = perform_basis_measurement([0.0, 1.0])
qc1.draw(output='mpl')
backend = Aer.get_backend('qasm_simulator')
job_sim = backend.run(transpile(qc1, backend), shots=5024)

# Grab the results from the job.
result_sim = job_sim.result()
counts = result_sim.get_counts(qc1)
plot_histogram(counts)
```

```
In [ ]:
# Next we try it on the |-> state
qc1 = perform_basis_measurement([1.0/np.sqrt(2), -1.0/np.sqrt(2)])
qc1.draw(output='mpl')
backend = Aer.get_backend('qasm_simulator')
job_sim = backend.run(transpile(qc1, backend), shots=5024)

# Grab the results from the job.
result_sim = job_sim.result()
counts = result_sim.get_counts(qc1)
plot_histogram(counts)
```

In [ ]:
```
#...and the |+> state
qc1 = perform_basis_measurement([1.0/np.sqrt(2), 1.0/np.sqrt(2)])
qc1.draw(output='mpl')
backend = Aer.get_backend('qasm_simulator')
job_sim = backend.run(transpile(qc1, backend), shots=5024)

# Grab the results from the job.
result_sim = job_sim.result()
counts = result_sim.get_counts(qc1)
plot_histogram(counts)
```

In [ ]:
```
#and now the cos(pi/8) |0> + sin(pi/8) |1> state
qc1 = perform_basis_measurement([np.cos(math.pi/8), np.sin(math.pi/8)])
qc1.draw(output='mpl')
backend = Aer.get_backend('qasm_simulator')
job_sim = backend.run(transpile(qc1, backend), shots=5024)

# Grab the results from the job.
result_sim = job_sim.result()
counts = result_sim.get_counts(qc1)
plot_histogram(counts)
```

Problem 2: EPR Pair Properties

Let's examine properties of the EPR pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,.$$

In what follows, let's suppose that Alice is given the left qubit of the EPR pair, and Bob is given the right qubit, and they are separated by a large distance.

a) Let $A = \{|a_1\rangle, |a_2\rangle\}$ be some orthonormal basis for $\mathbb{C}^2$. Suppose Alice measures her qubit using basis A. What are the statistics of the measurement outcomes (i.e. what are the probability of $|a_1\rangle$ or $|a_2\rangle$)?

Your Solution:

write your solution here, using LaTeX and Markdown

b) Show that if Alice obtains measurement outcome $|a_i\rangle$ for some $i \in \{1,2\}$, the post-measurement state of the EPR pair is $|a_i\rangle \otimes |a_i\rangle^*$ where $|a_i\rangle^*$ is the **complex conjugate** of $|a_i\rangle$ (i.e. the j-th entry is the complex conjugate of the j-th entry of $|a_i\rangle$).

This is interesting because Alice might have decided on the basis only after Bob was sent away, yet Alice's measurement causes Bob's qubit to instantaneously collapse into one of the basis states of A (up to complex conjugation). This is a phenomenon called

quantum steering, because Alice is able to **steer** Bob's qubit, even though she is only acting on **her** qubit.

Your Solution:

write your solution here, using LaTeX and Markdown

c) Suppose that Bob then measures his qubit using an orthonormal basis $B = \{|b_1\rangle, |b_2\rangle\}$. What are the statistics of his measurement outcomes, conditioned on Alice's outcome?

Your Solution:

write your solution here, using LaTeX and Markdown

d) Suppose the order of measurements were reversed: Bob measures his qubit first using basis B, and then Alice measures her qubit using basis A. Show that the **joint** probability distribution of their measurement outcomes is the same as before.

Your Solution:

write your solution here, using LaTeX and Markdown

e) What can you conclude about the effectiveness of using quantum entanglement and quantum steering as a method for faster-than-light communication? In other words, can Alice and Bob, by only making local measurements on their entangled state, send information to each other?

Your Solution:

write your solution here, using LaTeX and Markdown

Problem 3: Quantum Teleportation with Noise

We saw how to teleport quantum states in class. Let's consider a twist on the standard teleportation protocol. Let's imagine that when Alice and Bob meet up to create an entangled state, the settings on their lab equipment was screwed up and they accidentally create the following two-qubit entangled state

$$|\theta\rangle = \frac{1}{\sqrt{3}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle \ .$$

Only Alice realizes this after they haven each taken a qubit each and gone their separate ways.

Suppose that Alice now gets a gift qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Is there a way that she can still teleport $|\psi\rangle$ to Bob, using their corrupted entangled state $|\theta\rangle$ and the classical communication channel? Like in the standard teleportation protocol, Alice can only apply unitaries and measurements to her two qubits, and Bob will apply the same corrections as in the standard teleportation protocal (since he's not aware of the corruption).

a) Show how the teleportation protocol can be adapted for the corruption from Alice's side and analyze the correctness of your proposed protocol.

Your Solution:

write your solution here, using LaTeX and Markdown

b) Now let's implement Alice's teleportation protocol using the noisy EPR pair with qiskit.

Write code in `create_alice_noisy_tp_circuit` function below, which takes as as input a QuantumRegister (consisting of two qubits) and a ClassicalRegister (consisting of two 2 bits).

Important Note: the register indices in Alice's and Bob's functions are **local** (0-indexed), meaning that from Alice or Bob's point of view, her zeroth qubit is the gift qubit, and her first qubit is the first half of the EPR pair. From Bob's point of view, he only has the other half of the EPR pair, which he considers his zeroth qubit.

In [ ]:
```
def initialize_noisy_epr_pair(qc: QuantumCircuit, qubits: List[int]) -> Quar
    # For qc.initialize, the ordering of the states are |00>, |01>, |10>, |1
    #if the top wire corresponds to the rightmost bit (recall the little end
    qc.initialize([np.sqrt(1/3.0), np.sqrt(1/6.0), -np.sqrt(1/6.0), np.sqrt(
    qc.barrier()
    return qc

def create_base_noisy_tp_circuit() -> QuantumCircuit:
    qr1 = QuantumRegister(1, name="psi")
    qr2 = QuantumRegister(2, name="theta")
    cr = ClassicalRegister(2, name="m")
    qc = QuantumCircuit(qr1, qr2, cr)
    return initialize_noisy_epr_pair(qc, [1, 2])

def create_alice_noisy_tp_circuit(qr: QuantumRegister, cr: ClassicalRegister
    qc = QuantumCircuit(qr, cr)
    # Alice has two qubits (index 0,1) and access to two classical registers
    # ========= BEGIN CODE =================


    # ========= END CODE =================
    return qc
```

```
def create_bob_noisy_tp_circuit(qr: QuantumRegister, cr: ClassicalRegister)
    qc = QuantumCircuit(qr, cr)
    qc.z(0).c_if(cr[0], 1) # Apply gates if the registers
    qc.x(0).c_if(cr[1], 1) # are in the state '1'
    return qc
```

In [ ]:
```
noisy_tp_circuit = create_base_noisy_tp_circuit()
noisy_tp_circuit = append(noisy_tp_circuit, create_alice_noisy_tp_circuit, [
noisy_tp_circuit = append(noisy_tp_circuit, create_bob_noisy_tp_circuit, [2]
noisy_tp_circuit.draw(output='mpl')
```

In [ ]:
```
test_noisy_teleportation(noisy_tp_circuit)
```

Problem 4: Transferring Entanglement

Here we explore a task to **transfer entanglement**. Let's say there are three parties, Alice, Bob, and Carol. Alice shares an EPR pair with Bob, and Bob shares an EPR pair with Carol (so Alice has one qubit, Bob has two qubits, and Carol has one qubit).

a) Design and analyze a protocol that involves only classical communication between the pairs (Alice,Bob), and (Bob,Carol), such that at the end Alice and Carol --- who never directly interacted with each other --- now share an EPR pair.

Hint: use the teleportation protocol as inspiration.

Your Solution:

write your solution here, using LaTeX and Markdown

b) Now let's implement Alice's, Bob's and Carol's parts of the entanglement transferring circuit. You will have to implement what Alice, Bob, and Carol do with their qubits, and how they classically communicate with each other. Fill in the functions in the places indicated below.

Important note: see the important note in Problem 3 regarding the local indexing of qubits in the Alice, Bob and Carol functions.

In [ ]:
```
def alice_circuit(qr: QuantumRegister, cr: ClassicalRegister) -> QuantumCirc
    qc = QuantumCircuit(qr, cr)
    # Alice has one qubit (index 0) and access to two classical registers (i
    # ========= BEGIN CODE =================



    # ========= END CODE =================
    return QuantumCircuit(qr, cr)

def bob_circuit(qr: QuantumRegister, cr: ClassicalRegister) -> QuantumCircui
```

```
    qc = QuantumCircuit(qr, cr)
    # Bob has two qubits (index 0,1) and access to four classical registers
    # ========= BEGIN CODE =================

    # ========= END CODE =================
    return qc

def carol_circuit(qr: QuantumRegister, cr: ClassicalRegister) -> QuantumCirc
    qc = QuantumCircuit(qr, cr)
    # Carol has one qubit (index 0) and access to two classical registers (i
    # look up qiskit documentation for using classical registers to control
    # ========= BEGIN CODE =================


    # ========= END CODE =================
    return qc

def add_epr_pair(qc: QuantumCircuit, a, b):
    qc.h(a)
    qc.cnot(a,b)
    qc.barrier()
    return qc

def create_entanglement_transfer_circuit_base() -> QuantumCircuit:
    """
    This creates a circuit with 2 EPR pairs in registers {0, 1} and {2, 3} r
    and four classical registers (labelled {0,1,2,3}).
    Alice will have access to qubit 0, and the first two classical registers
    Bob will have access to qubits 1 and 2, and all the classical registers
    Carol will have access to qubit 3, and the last two classical registers
    """
    qr1 = QuantumRegister(2, name="epr ab")
    qr2 = QuantumRegister(2, name="epr bc")

    num_classical_bits = 4

    cr = ClassicalRegister(num_classical_bits, name="cr")
    global_circuit = QuantumCircuit(qr1, qr2, cr)
    global_circuit = add_epr_pair(global_circuit, 0, 1)
    global_circuit = add_epr_pair(global_circuit, 2, 3)
    return global_circuit

def create_entanglement_transfer_circuit() -> QuantumCircuit:
    qc = create_entanglement_transfer_circuit_base()

    qc = append(qc, alice_circuit, [0],[0,1])
    qc.barrier()
    qc = append(qc, bob_circuit, [1, 2], [0, 1,2,3])
    qc.barrier()
    qc = append(qc, carol_circuit, [3], [2, 3])

    return qc
```

In [ ]:

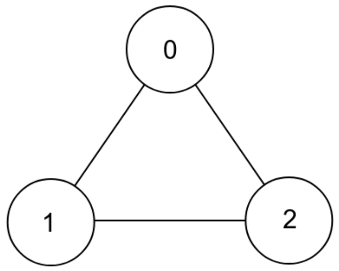
```
entanglement_transfer_circuit = create_entanglement_transfer_circuit()
entanglement_transfer_circuit.draw(output = 'mpl')
```

In [ ]:
```
test_entanglement_transfer(entanglement_transfer_circuit)
```

Problem 5: Let's Play a (Nonlocal) Game

In class we learned about the CHSH game, let's consider another game, called *Let's Confuse Charlie*.

The game is inspired by the following simple fact: the vertices in the following graph cannot be colored red or blue such that adjacent vertices have different colors. In other words, the triangle is not 2-colorable.



Nonetheless, Alice and Bob decide to make a game to convince a third party, called Charlie, that it *is* possible to 2-color a triangle. (They have a lot of time on their hands, clearly). Consider the following game:

1. Charlie picks a vertex s in the graph (0, 1, or 2) uniformly at random, and sends s to Alice.
2. Charlie picks a t that is either s or $s+1 \mod 3$ with $\frac{1}{2}$ probability each, and then sends t to Bob.
3. Alice and Bob have to give colors "red" or "blue" back to Charlie.
4. If $s = t$, then Alice and Bob win if they give the same color. If $s \neq t$, then Alice and Bob win if they have different colors.

a) What is the best probability that any local hidden variable (LHV) strategy for Alice and Bob can win this game?

Your Solution:

write your solution here, using LaTeX and Markdown

b) In this part you will come up with a *quantum strategy* for Alice and Bob to beat the LHV (i.e. classical) winning probability. You will code up a strategy by specifying a measurement for Alice that depends on her question s, and a measurement for Bob that depends on his question t. We assume that Alice and Bob share a single EPR pair

$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$. So in "alice_game_circuit" for example, you will write based on Alice's question s (which is 0, 1 or 2) what single-qubit gates Alice performs on her qubit using the qc object (which has a single qubit), and then finally Alice should perform a measurement on her qubit to obtain a 0 or 1 (which represents the reported color of her vertex). You will do a similar thing for Bob.

Try to get the best winning probability you can.

Hint: use the CHSH strategy for inspiration.

```
In [ ]: def alice_game_circuit(qr: QuantumRegister, cr: ClassicalRegister, s: int) -
            qc = QuantumCircuit(qr, cr)
            # Alice will need to apply a unitary gate, and then measure her qubit
            # ========= BEGIN CODE =================

            # ========= END CODE =================
            return qc

        def bob_game_circuit(qr: QuantumRegister, cr: ClassicalRegister, t: int) ->
            qc = QuantumCircuit(qr, cr)
            # Bob will need to apply a unitary gate, and then measure his qubit
            # ========= BEGIN CODE =================

            # ========= END CODE =================
            return qc
```

The following code will test how well your strategy performs.

```
In [ ]: def play_game(s: int, t: int) -> float:
            hidden_state = QuantumRegister(2, name="epr_pair")
            answers = ClassicalRegister(2, name="answer")
            global_circuit = QuantumCircuit(hidden_state, answers)
            global_circuit = add_epr_pair(global_circuit, 0, 1)
            global_circuit = append(global_circuit, lambda qr, cr : alice_game_circu
            global_circuit = append(global_circuit, lambda qr, cr : bob_game_circuit
            total_shots = 5024
            backend = Aer.get_backend('qasm_simulator')
            job_sim = backend.run(transpile(global_circuit, backend), shots=total_sh
            result_sim = job_sim.result()
            measurements = result_sim.get_counts(global_circuit)
            winning_shots = 0
            if s == t:
                for measurement in measurements:
                    if measurement[0] == measurement[1]:
                        # Win this game
                        winning_shots += measurements[measurement]
            else:
                for measurement in measurements:
                    if measurement[0] != measurement[1]:
                        # Win this game
                        winning_shots += measurements[measurement]
            return winning_shots / total_shots
```

```
winning_probability = 0.0
for i in range(3):
    winning_probability += play_game(i, i)
    winning_probability += play_game(i, (i + 1) % 3)
print("Average Winning Probability: ", winning_probability / 6)
```

c) Describe the strategy that you chose, and algebraically compute its winning probability (i.e. carry out a calculation similar to how the quantum strategy for CHSH was analyzed in class), and confirm it matches the numerics above.

Your Solution:

write your solution here, using LaTeX and Markdown

BONUS PROBLEM If you think you have the optimal quantum strategy for this game, give a proof that there is no better quantum strategy. You may assume Alice and Bob use 1 EPR pair as their shared state.

Your Solution:

write your solution here, using LaTeX and Markdown

```
In [ ]:
```